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The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1938 and reported to the Secretary.

Allegheny Mountain.
Illinois, Carbondale, May 13–14.
Indiana, Terre Haute, May 6–7.
Iowa, Sioux City, April 15–16.
Kansas, Pittsburg, April 2.
Kentucky, Morehead, May 14.
Louisiana-Mississippi, Starkville, Miss., March 11–12.
Maryland-District of Columbia-Virginia, Annapolis, Md., May 7.
Michigan, Ann Arbor, March 19.
Minnesota, Collegeville, May 14.
Missouri.
Nebraska, Hastings, May 6.
Ohio, Columbus, March 31.
Oklahoma, Oklahoma City, February 11.
Philadelphia, Collegeville, Pa., Nov. 26
Rocky Mountain, Boulder, Colo., April 15–16.
Southeastern, Atlanta, Ga., April 1–2.
Southern California, Claremont, March 26.
Southwestern, Albuquerque, N.M., April 25–26.
Texas, April.
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ON CONVEXITY*

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1. Introduction. At least two different approaches to an intuitional notion of convexity are common in elementary texts. A convex closed curve is described as one which can be cut by a straight line in only two points. But a convex polygon is described as one no extended side of which will pass through the interior of the polygon. A pupil or teacher, seeking a *logical* justification for the use of the same adjective *convex* in the two instances, might have some difficulty. Justification lies of course in the fact that curve and polygon alike bound convex sets of points. It happens that one characteristic property of such sets suggests the description cited for the convex curve, while a different property suggests the description of the convex polygon.

In this paper attention will be called to certain properties, relative to point sets, which seem to be inherent in any intuitional notion of convexity, and consideration given to the precise interrelationships of these properties. For simplicity, the discussion will be restricted to point sets in a plane; but it may be said in advance that it all applies, with only the most obvious alterations, to point sets in euclidean space of n dimensions. In fact some parts are extensible to more general spaces, as will be indicated near the end of the paper.

Critical study of convex sets seems to have originated with Minkowski [1].† A simplified and elegant treatment of some of Minkowski's ideas was given by Carathéodory [2]. By far the most comprehensive treatment is that by Bonnesen and Fenchel [3], in which is to be found an extensive bibliography to 1934.

2. Five properties which point sets may possess. A point set *may* be

1. Bounded
2. Closed
3. Possessed of inner points
4. Linearly connected
5. Completely supported at its boundary points.

A set is *bounded* if there exists a definite square within which all of its points lie. In analytic terms, there exists a positive constant M such that for each point (x, y) of the set, $|x| < M$ and $|y| < M$.

A set is *closed* if it contains all of its limit points. That is, if $\{(x_n, y_n)\}$ is a sequence of points of the set, and if $\lim_{n\to\infty} (x_n, y_n) = (\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y})$ belongs to the set.

Inner points, the possession of which constitutes property 3, comprise one of three mutually exclusive categories into which any well defined point set divides the points in its plane. Relative to a given set a point will be called: an *exterior* point if it does not belong to the set, a *boundary* point if it belongs to the set but is a limit point of exterior points, an *inner* point if it belongs to the set and is not a boundary point.*

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* This paper includes the substance of an address delivered by invitation before the Mathematical Association of America, Dec. 31, 1937, at Indianapolis. Presented for the Slaught Memorial Volume of the MONTHLY.

† Numbers in brackets refer to references at the end of the paper.

A set is *linearly connected* if membership of points A and B in the set implies membership of all points on the line segment connecting A and B.†

A definition of the fifth property must be preceded by some preliminary discussion regarding the relation of a point set to the lines in its plane. Any line divides a plane in which it lies into two half-planes which may be designated as *closed* or *open* according as they are thought of as including or excluding the points of the line itself. Analytically, the line

$$l(x, y) \equiv ax + by + c = 0$$

determines two closed half-planes, one consisting of those points (x, y) for which $l(x, y) \geqq 0$, the other of those for which $l(x, y) \leqq 0$. The corresponding open half-planes are represented if the equality signs be omitted.

Any point set $\mathfrak{M}$ serves to classify all lines in its plane into three categories with associated names as follows. Relative to the point set $\mathfrak{M}$, a line l is called: (1) a *bounding* line if all points of $\mathfrak{M}$ lie in one of the open half-planes determined by l, (2) a *separating* line if points of $\mathfrak{M}$ lie in each of the open half-planes, or (3) a *supporting* line if all points of $\mathfrak{M}$ lie in one of the closed half-planes and at least one point of $\mathfrak{M}$ lies on l.‡

A set of points is *completely supported at its boundary* if through each of its boundary points there can be passed a supporting line relative to the set.

3. Fundamental theorem on convexity. The theorem to be considered in this section justifies the two different intuitional approaches to the notion of convexity mentioned in our introduction. It was first proved by Minkowski, but his proof was scattered over many pages. Other proofs have since been given by various authors, the one here presented being due essentially to Carathéodory.

THEOREM I. *For a set of points which is bounded and closed and possesses inner points, the two properties linearly connected and completely supported at the boundary are logically equivalent.*

Let $\mathfrak{M}$ be a set of points which is bounded and closed and possesses inner points.

We will first assume that it is linearly connected, and prove that it is completely supported at the boundary by establishing the following two propositions:

---

* It is to be noted of course that, for a particular point set, one or more of these categories may be vacant. For example the set of points on a line segment has no inner points (in the sense defined), and hence does not possess property 3.

† This property is usually taken alone as characteristic of a convex set. Thus the set consisting of *all* points in a plane would be said to be convex. There would seem to be at least an intuitional advantage in the more directly descriptive term *linearly connected.*

‡ For example, a side of an elementary convex polygon is a segment of a supporting line relative to the set of points within and on the polygon. Likewise, a tangent to a circle is a supporting line relative to the set of points within and on the circle.

(A) Through each exterior point there passes a bounding line of $\mathfrak{M}$.

(B) From a sequence of bounding lines passing respectively through a sequence of exterior points which has a boundary point B of $\mathfrak{M}$ as limit point, there can be chosen a subsequence of lines, the limiting (in a sense to be defined) line of which is a supporting line of $\mathfrak{M}$ through B.

To prove (A), let E be any exterior point relative to $\mathfrak{M}$. Then, since $\mathfrak{M}$ is bounded and closed, and since the distance from E to a variable point (x, y) of $\mathfrak{M}$ is a continuous function of x and y, this distance has, by a well known theorem of Weierstrass, a definite minimum which is attained at some point M of $\mathfrak{M}$. The line through E, perpendicular to the segment EM is then a bounding line for $\mathfrak{M}$. For otherwise it would either contain or separate points of $\mathfrak{M}$; indeed it would certainly *contain* a point of $\mathfrak{M}$ since $\mathfrak{M}$ is linearly connected. But if it contained a point Q of $\mathfrak{M}$, then all points of the segment MQ would belong to $\mathfrak{M}$, and consideration of the right triangle MEQ shows that M would not be the nearest point of $\mathfrak{M}$ to E.

To prove (B), let the boundary point $B=(\bar{x}, \bar{y})$ of $\mathfrak{M}$ be the limit of a sequence of exterior points $\{E_n\} = \{(x_n, y_n)\}$, and let

$$(1_n) \qquad a_n x + b_n y + c_n = 0$$

be the equation of a bounding line through E_n. We can and will assume that these equations are in normal form, so that $a_n^2 + b_n^2 = 1$ and $|c_n|$ is the distance of the line (1_n) from the origin. Since the sequence of number triples $\{(a_n, b_n, c_n)\}$ is bounded, there exists a number triple (a, b, c) which is the limit of some infinite subsequence

$$(2) \qquad \{(a_\nu, b_\nu, c_\nu)\}, \qquad (\nu = n_1, n_2, n_3, \cdots),$$

of $\{(a_n, b_n, c_n)\}$. The triple (a, b, c) serves to determine a line

$$(3) \qquad ax + by + c = 0$$

which is a supporting line of $\mathfrak{M}$ through B.

If the last statement is not obvious, it may be verified analytically by consideration of the function

$$f(u, v, w, x, y) \equiv ux + vy + w$$

of the five variables u, v, w, x, y. Since f is continuous, the limiting value of $f(u, v, w, x, y)$, as $(u, v, w, x, y) \to (a, b, c, \bar{x}, \bar{y})$, is $f(a, b, c, \bar{x}, \bar{y})$. But if the approach is made over the sequence of points $\{(a_\nu, b_\nu, c_\nu, x_\nu, y_\nu)\}$, the value of the function is always zero, since the point $E_\nu = (x_\nu, y_\nu)$ is on the line (1_ν). Therefore $f(a, b, c, \bar{x}, \bar{y}) = 0$; that is, the line (3) passes through B.

Furthermore the line (3) does not separate points of $\mathfrak{M}$. For if it did there would be points (x', y') and (x'', y'') of $\mathfrak{M}$, such that $f(a, b, c, x', y')$ and $f(a, b, c, x'', y'')$ would have different signs; hence from the continuity of f and the property of the sequence (2), there would be a value of ν for which

$f(a_\nu, b_\nu, c_\nu, x', y')$ and $f(a_\nu, b_\nu, c_\nu, x'', y'')$ would have different signs. But this would imply that the line (1_ν) was a separating line of $\mathfrak{M}$ and so contradict our assumption regarding it. Hence the line (3) is indeed a supporting line of $\mathfrak{M}$ through B.

Conversely, if $\mathfrak{M}$ is completely supported at the boundary it is linearly connected, the proof being relatively simple.

To prove it indirectly, we suppose M and N are points of $\mathfrak{M}$ while a point E on the segment MN does not belong to $\mathfrak{M}$. We join E to any inner point I of $\mathfrak{M}$ not on the line MN. Since E is an exterior point and I an inner point, there must be on the segment EI a boundary point of $\mathfrak{M}$; call it B. By hypothesis there passes through B a supporting line s of $\mathfrak{M}$. And by definition all points of $\mathfrak{M}$ must lie in the same s-determined half-plane, which half-plane must also contain E since it is on the segment MN. But this is impossible, since s certainly separates I and E. The contradiction proves that E must belong to $\mathfrak{M}$, that is, $\mathfrak{M}$ is linearly connected.

It is to point sets which are bounded, closed, possess inner points, and have either (and hence both) of the properties linearly connected and completely supported at the boundary, that the term *convex* can be applied without doing violence to elementary intuitional notions. The boundary points of such a set constitute a convex curve (possibly a polygon). The analog of such a set in space of three or more dimensions is termed a *convex body* and its boundary points constitute a *convex surface*.

4. Two stronger theorems. While Theorem I is pleasing because of its symmetry and its adequacy in the case of intuitionally convex sets, it is not the strongest theorem which can be stated relative to the properties in question. On considering the two halves of the theorem separately and critically, we find that the hypotheses can be weakened without affecting the conclusions. Thus we obtain the two theorems:

THEOREM II. *If a point set is linearly connected it is completely supported at its boundary points.*

THEOREM III. *If a point set is closed and possesses inner points and is comcompletely supported at its boundary points, it is linearly connected.*

With reference to Theorem II, we note first that if the point set in question possesses no boundary points it is vacuously true, and if the set consists of a single point it is trivially true. Suppose then that the linearly connected set $\mathfrak{M}$ has a boundary point B and at least one other point M. We denote by $\mathfrak{M}_0$ the subset consisting of those points of $\mathfrak{M}$ which lie within or on the circle with center B and radius BM, and by $\overline{\mathfrak{M}}_0$ the closure of $\mathfrak{M}_0$, that is, the set $\mathfrak{M}_0$ augmented by all of its limit points. Then the set $\overline{\mathfrak{M}}_0$ satisfies the hypothesis of Theorem I except possibly for the possession of inner points, and no use was made of this latter property in the proof of the first half of Theorem I. Furthermore B is a boundary point of $\overline{\mathfrak{M}}_0$, and so we may conclude, as in Theorem I, that through B there passes a supporting line s of $\overline{\mathfrak{M}}_0$. But then s must also be a supporting line of the original set $\mathfrak{M}$. For if there were points Q and R of $\mathfrak{M}$, one in each of the open half-planes determined by s, all points of the segment BQ would lie in one of the half-planes and those of the segment BR in the other; so the line s would certainly separate points of $\overline{\mathfrak{M}}_0$, contradicting its character as a supporting line of that set.

Theorem III is merely the second half of Theorem I with the assumption of boundedness omitted from the hypothesis. Since no use was made of that property in the proof of that half, the proof is adequate for Theorem III.

5. Complete existential theory of properties 1–5. The five properties listed in §2 are clearly not independent. It is perhaps natural to ask if Theorems II and III express the only implicational relationships between them. The question can be answered only after an exploration of all possibilities. Such an exploration relative to a set of properties leads to what E. H. Moore called their *complete existential theory*. In the case of our five properties the question is the following. Of the $2^5=32$ composite properties representable by the notations

$$(+++++),\ (-++++),\ \cdots,\ (----+),\ (-----),$$

the ith symbol within parentheses being $+$ or $-$ according as property i is present or absent in the composite property, which of them can be possessed by existent (non-null) point sets?

Theorem II eliminates the possibility of eight combinations characterized by $(\cdot\cdot\cdot+-)$, and Theorem III eliminates the possibility of two characterized by $(\cdot++-+)$. The remaining twenty-two composite properties are possessed by actually existent point sets, as is shown by the examples listed below. Hence Theorems II and III express the only significant implicational relationships between the properties 1–5. For convenience in writing, the sets are described analytically by conditions upon the coördinates (x, y) of points constituting them. The examples are all simple and can be easily visualized or represented graphically and their characters verified.*

Character	Point set
1. $(+++++)$	$x^2+y^2\leqq 1$
2. $(-++++)$	(x, y) unrestricted
3. $(+-+++)$	$x^2+y^2<1$
4. $(++-++)$	$0\leqq x\leqq 1,\ y=0$
5. $(--+++)$	$x>0,\ y>0$
6. $(-+-++)$	$x\geqq 0,\ y=0$
7. $(+--++)$	$0<x<1,\ y=0$
8. $(+-+-+)$	$x^2+y^2<1$; and $(2, 0)$
9. $(++--+)$	$(0, 0)$; and $(1, 0)$
10. $(+++--)$	$1\leqq x^2+y^2\leqq 2$

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* However it may be more amusing and profitable for the reader (with time and interest) to construct his own examples. The search for examples emphasizes the rôles played by the individual properties in our intuitional notion of convexity.

11.	$(++---)$	$x^2+y^2=1$; and $(0, 0)$
12.	$(+-+--)$	$x^2+y^2\leqq 1$; and $1\leqq x<2$, $y=0$
13.	$(+---+)$	$(0, 0)$; and $1<x\leqq 2$, $y=0$
14.	$(-++--)$	$x^2+y^2\leqq 1$; and $x\geqq 1$, $y=0$
15.	$(-+--+)$	$(0, 0)$; and $x\geqq 1$, $y=0$
16.	$(--+-+)$	$(0, 0)$; and $x>1$, y unrestricted
17.	$(---++)$	$x>0$, $y=0$
18.	$(+----)$	$x^2+y^2=1$; and $1\leqq x<2$, $y=0$; and $2<x\leqq 3$, $y=0$
19.	$(-+---)$	$x^2+y^2=1$; and $x\geqq 1$, $y=0$
20.	$(--+--)$	$x^2+y^2\leqq 1$; and $1\leqq x<2$, $y=0$; and $x>2$, $y=0$
21.	$(----+)$	$(0, 0)$; and $x>1$, $y=0$
22.	$(-----)$	$x^2+y^2=1$; and $1\leqq x<2$, $y=0$; and $x>2$, $y=0$.

6. The extension of a set to attain a specified property. Of the five properties listed in §2, properties 2 and 4 are *closure* properties in the sense that they assure, in a set which possesses them, the inclusion of certain subsets of points determined by points of the set and by the property in question. Relative to such a property it is pertinent to ask whether a set which does not have it can be *extended*, by the adjunction of points, so that the extended set will have the property. And if so, what is the least extensive set, containing the given set, which will have the desired property?

Obviously the *least extensive* set containing a set $\mathfrak{M}$ and having a property P must be contained in every such set. It could therefore be nothing other than the greatest common subset of all such sets. If this greatest common subset *has*[*] the property P, it is the desired extended set. It may suitably be called the *P-extension of* $\mathfrak{M}$ and denoted by $\mathfrak{M}_P$. Thus $\mathfrak{M}_2$, the closed extension of $\mathfrak{M}$, is the greatest common subset of all the closed sets containing $\mathfrak{M}$. It is generally known as the closure of $\mathfrak{M}$. And $\mathfrak{M}_4$, the linearly connected extension of $\mathfrak{M}$, is the greatest common subset of all the linearly connected sets containing $\mathfrak{M}$. The composite property, closed and linearly connected, leads to the extension $\mathfrak{M}_{24}$, the greatest common subset of all the closed and linearly connected sets containing $\mathfrak{M}$.

The extensions $\mathfrak{M}_4$ and $\mathfrak{M}_{24}$ have been of use in many connections. A few remarks as to the terminology usually associated with them may be in order. Earlier writers, Carathéodory and F. Riesz for example, merely referred to them as the smallest convex (or smallest closed convex) set containing the given set. Later writers have almost universally adopted the term "convex hull" (konvexe Hülle). Brevity and suggestive terminology are of course appreciated by everyone. But terminology which suggests the *wrong* idea is surely undesirable. The term "hull" probably suggests to most English-speaking persons an *outside covering*, hence it might most appropriately be applied to the aggregate of *boundary* points of a convex set, an aggregate quite different from that denoted by $\mathfrak{M}_4$*.

---

[*] It should be noted that the greatest common subset of all sets containing $\mathfrak{M}$ and having a given property P does not necessarily have the property P. As an illustration, one may take for $\mathfrak{M}$ the set of points on a line segment and for P the property of possessing inner points.

From Theorem II it follows that $\mathfrak{M}_4$ is completely supported at its boundary points. And it is easily seen that if $\mathfrak{M}$ is bounded, closed, or possesses inner points, the same respective properties persist in $\mathfrak{M}_4$.

If all points of $\mathfrak{M}$ lie on a single line, then $\mathfrak{M}_4$ lies on that line; and if in addition $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ is a closed line segment.

If $\mathfrak{M}$ consists of a finite number of points not on a line, then $\mathfrak{M}_4$ is the set of points within and on the largest polygon having points of $\mathfrak{M}$ for vertices.

7. Two characteristic properties of the linearly connected extension of a bounded and closed set.

THEOREM IV. *If* $\mathfrak{M}$ *is bounded and closed,* $\mathfrak{M}_4$ *consists precisely of those points through which pass no bounding lines of* $\mathfrak{M}$.

First, if a point A belongs to $\mathfrak{M}_4$ it cannot be on any bounding line l of $\mathfrak{M}$. For if B is a point of $\mathfrak{M}$ at minimum distance from the bounding line l, the line s through B parallel to l is a supporting line of $\mathfrak{M}$. Since all points of $\mathfrak{M}$ lie in that one of the s-determined closed half planes which does not contain l, the same must be true of $\mathfrak{M}_4$, from its definition. Hence A cannot lie on l.

Conversely, if A does not belong to $\mathfrak{M}_4$, there passes through it a bounding line of $\mathfrak{M}_4$ which can be determined by the method indicated in the proof of Theorem I. And a bounding line of $\mathfrak{M}_4$ is *a fortiori* a bounding line of $\mathfrak{M}$.

A neat and useful analytic expression can be given to the content of Theorem IV. The linear equation

$$u(x - x_0) + v(y - y_0) = 0$$

with u and v arbitrary, represents the family of lines through the point (x_0, y_0). A line of this family will be a bounding line of a given set of points $\mathfrak{M}$, if and only if for some number pair (u, v) the inequality

$$u(x - x_0) + v(y - y_0) > 0 \tag{5}$$

is satisfied for all points (x, y) of $\mathfrak{M}$. Hence if $\mathfrak{M}$ is bounded and closed, a necessary and sufficient condition that (x_0, y_0) belong to $\mathfrak{M}_4$ is that the system of linear inequalities represented by (5) when (x, y) varies over the set $\mathfrak{M}$ shall admit no simultaneous solution (u, v). This general idea has been used in the study of linear inequalities by Kakeya, Haar, Fujiwara, Stokes, Motzkin, Dines, and McCoy.

THEOREM V. *If* $\mathfrak{M}$ *is bounded and closed,* $\mathfrak{M}_4$ *consists precisely of those points each of which can be the center of mass of a distribution of positive masses (of total mass unity)*† *at suitable points of* $\mathfrak{M}$.

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* In a recent paper (Bulletin of the American Mathematical Society, 1936, p. 353), the present writer used the term "convex extension" instead of convex hull. Many persons with whom he has discussed the matter agree that the latter term is inappropriate, but likewise agree that it is firmly intrenched in the mathematical vocabulary.

† The parenthetic phrase has no particular significance here. The possibility of taking the total mass unity is useful in the analytic statement of the theorem (Corollary II, below).

For the proof we need the following lemma, which is however of some importance in itself.

Lemma. *If* $\mathfrak{M}$ *is bounded and closed, and s is a supporting line of* $\mathfrak{M}_4$*, then the set of points in which s intersects* $\mathfrak{M}_4$ *is precisely the linearly connected extension of the set in which s intersects* $\mathfrak{M}$.

If we denote the set of all points on s by $\mathfrak{S}$, those common to $\mathfrak{S}$ and $\mathfrak{M}$ by $\mathfrak{S}\mathfrak{M}$, and those common to $\mathfrak{S}$ and $\mathfrak{M}_4$ by $\mathfrak{S}\mathfrak{M}_4$, the assertion of the lemma may be written

$$\mathfrak{S}\mathfrak{M}_4 \equiv (\mathfrak{S}\mathfrak{M})_4. \tag{6}$$

That the left side of (6) includes the right follows immediately. For since $\mathfrak{M}_4$ includes $\mathfrak{M}$, $\mathfrak{S}\mathfrak{M}_4$ includes $\mathfrak{S}\mathfrak{M}$. And since $\mathfrak{S}\mathfrak{M}_4$ is linearly connected, it must contain $(\mathfrak{S}\mathfrak{M})_4$ by the definition of the latter set.

To prove that conversely the left side of (6) is included in the right we proceed indirectly, assuming that a point P belongs to $\mathfrak{S}\mathfrak{M}_4$ but not to $(\mathfrak{S}\mathfrak{M})_4$, and obtaining therefrom a contradiction. To facilitate the discussion we introduce a system of coördinate axes with the origin (0, 0) at P and the x-axis along the supporting line s and so directed that for all points of $\mathfrak{M}_4$, and hence of $\mathfrak{M}$, $y \geqq 0$. At least one point of $\mathfrak{M}$ is on the x-axis; for otherwise s would be a bounding line of $\mathfrak{M}$, and (by Theorem IV) P could not belong to $\mathfrak{M}_4$. However, since $(\mathfrak{S}\mathfrak{M})_4$ is linearly connected and is assumed not to contain P, all points of $\mathfrak{S}\mathfrak{M}$ must lie on one side of P; we will suppose for definiteness that, for all points of $\mathfrak{S}\mathfrak{M}$, $x<0$.

But for some points of $\mathfrak{M}$, $x \geqq 0$; otherwise the y-axis, through P, would be a bounding line of $\mathfrak{M}$. Let us denote the bounded and closed subset of $\mathfrak{M}$ in the first quadrant by $\mathfrak{M}_I$, the minimum distance of $\mathfrak{M}_I$ from the x-axis by h,* and the maximum distance of $\mathfrak{M}_I$ from the y-axis by k. Then the line through $P(0, 0)$ with slope $h/2k$ is certainly a bounding line of $\mathfrak{M}$. Hence (by Theorem IV) P cannot belong to $\mathfrak{M}_4$. *A fortiori P* cannot belong to $\mathfrak{S}\mathfrak{M}_4$ as we assumed, and the lemma is established.

Now we are in a position to prove Theorem V. And we first show that if P belongs to $\mathfrak{M}_4$, it is the center of mass of a suitable distribution of masses, of total mass unity, at points of $\mathfrak{M}$. In the simplest case where $\mathfrak{M}$ is on a single line, say on the x-axis, $\mathfrak{M}_4$ consists, as has been noted, of the points

$$x = ta + (1 - t)b, \qquad (0 \leqq t \leqq 1),$$

of a line segment joining the extreme points $(a, 0)$ and $(b, 0)$ of $\mathfrak{M}$. If P is a point of this segment with abscissa

$$\bar{x} = \bar{t}a + (1 - \bar{t})b,$$

then P is the center of mass of a distribution $\bar{t}$ at $(a, 0)$ and $(1-\bar{t})$ at $(b, 0)$.

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* Certainly *positive* in view of our supposition that for all points of $\mathfrak{S}\mathfrak{M}$, $x<0$.

If $\mathfrak{M}$ does not lie entirely on one line and P is any point of $\mathfrak{M}_4$, we may join any point M of $\mathfrak{M}$ to P and extend (if necessary) the segment MP to intersect the boundary of $\mathfrak{M}_4$ at a point B. If B is a point of $\mathfrak{M}$ we may conclude immediately that P is the center of mass of suitable masses at B and M. If B is not a point of $\mathfrak{M}$ we make use of the supporting line s of $\mathfrak{M}_4$ through B. By the lemma, B belongs to $(\mathfrak{S}\mathfrak{M})_4$ which must be a line segment of s terminated by points Q and R of $\mathfrak{M}$. Hence B is the center of mass of a suitable distribution of masses at Q and R. These masses may be μ_2 and μ_3, such that $\mu_2+\mu_3=1$, or they may be any two masses proportional to μ_2 and μ_3. If the proportionality factor be chosen properly, a complementary mass may be placed at M so that the sum of the three masses is unity and the center of mass is P. The coördinates $(\bar{x}, \bar{y})$ of P may be expressed explicitly in terms of the coördinates of $M(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ as follows. The coördinates of B (on the segment QR) may be expressed in the form $(\mu_2x_2+\mu_3x_3,\ \mu_2y_2+\mu_3y_3)$; and hence those of P (on the segment MB) may be expressed in the form

$$\bar{x} = \bar{t}x_1 + (1-\bar{t})(\mu_2x_2 + \mu_3x_3), \qquad \bar{y} = \bar{t}y_1 + (1-\bar{t})(\mu_2y_2 + \mu_3y_3)$$

for some value of $\bar{t}$ between 0 and 1. So that, since $\mu_2+\mu_3=1$, the coördinates $(\bar{x}, \bar{y})$ of P may be written

$$\bar{x} = m_1x_1 + m_2x_2 + m_3x_3, \qquad \bar{y} = m_1y_1 + m_2y_2 + m_3y_3,$$

where m_1, m_2, and m_3 are positive and $m_1+m_2+m_3=1$.

Conversely, if P is the center of mass of positive mass distribution at points of $\mathfrak{M}$, then P belongs to $\mathfrak{M}_4$. For otherwise there would exist through P (by Theorem IV) a bounding line of $\mathfrak{M}$. Since all points of $\mathfrak{M}$ would lie on one side of this line, P could not be the center of mass. Analytically, if P were the origin (0, 0), and the bounding line the y-axis, the zero abscissa of P could not possibly be given by the formula

$$\bar{x} = \sum_{i=1}^{r} m_ix_i \Big/ \sum_{i=1}^{r} m_i, \qquad m_i > 0,$$

or by the analogous integral formula for the center of mass of a continuous mass distribution.

The following two corollaries emerge immediately from the preceding proofs:

COROLLARY I. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists of the points of $\mathfrak{M}$, together with the points of all line segments which join points of $\mathfrak{M}$, and the points of all triangular regions with vertices belonging to $\mathfrak{M}$.*

COROLLARY II. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists of those points whose coördinates can be expressed in the form*

$$x = m_1x_1 + m_2x_2 + \cdots + m_rx_r,$$
$$y = m_1y_1 + m_2y_2 + \cdots + m_ry_r,$$

where (x_1, y_1), (x_2, y_2), $\cdots$, (x_r, y_r) *are points of* $\mathfrak{M}$; m_1, m_2, $\cdots$, m_r *are non-negative; and* $\sum_{i=1}^{r} m_i = 1$. *Furthermore, if such expression is possible for* (x, y), *it is possible with* $r \leqq 3$.

The analytic criteria in Corollary II (and in its generalization to n dimensions) have been useful in many applications. The most obvious is of course the study of non-negative solutions of systems of linear equations.

8. Generalizations. It has already been remarked that the theory presented in the preceding sections generalizes immediately to n-dimensional euclidean space. The two-dimensional space of points $P(x, y)$ is replaced by the space of points $P(x_1, x_2, \cdots, x_n)$, the lines $ax+by+c=0$ by hyperplanes $a_1x_1+a_2x_2+\cdots+a_nx_n+c=0$, the half-planes by half-spaces, and so on.

It is natural to inquire whether further generalization is possible. In present day mathematics, the term *space* has become almost synonymous with a set of elements of any sort, and adjectives are prefixed to indicate the nature of such special properties as the set is assumed to possess. Thus we speak of metric space, linear or vector space, topological space, function space, Hilbert space, Banach space, and so on.

Without attempting any detailed or comprehensive study we may perhaps inquire as to the properties which a space (set of elements) should have in order that a generalization of the elementary notion of convexity may have significance in the space.

Our property 1 (boundedness) immediately suggests that the space should be a metric space, that it should have associated with it a distance function adequate to determine a definite distance between each pair of elements, with the usual properties of distance. If the metric space be complete (that is contain a limit element associated with each convergent sequence of elements), the properties 2 (closure) and 3 (possession of inner points) can be immediately generalized. Direct generalization of property 4 (linearly connectedness) would seem to require the notion of line in the space, that is, essentially a linear or vector space. A well known space which furnishes all the desiderata mentioned with a minimum of postulates is the so-called Banach space [**4,** p. 53]. And so it is not surprising to find recent papers [**5, 6, 7, 8**] dealing with convexity in Banach space.

But the most obvious is not the only method of generalizing. The property of linearly connectedness, as we have defined it for sets in euclidean space seems to depend upon the presence of lines in the space. Nevertheless Menger [**9**] has defined, relative to sets in a space which is merely assumed to be metric, a property which he calls convexity, and which upon specialization to euclidean space turns out to be entirely equivalent to linearly connectedness.

Generalization of property 5 (complete support at the boundary) calls for generalization of the euclidean hyperplane. This can easily be secured by introduction of a generalized inner product between elements of a Banach space, generalizing the ordinary inner product $a \cdot x \equiv a_1x_1+a_2x_2+\cdots+a_nx_n$ of two euclidean vectors. But again the obvious way is not the only way. For Banach, on the last page of his book, sketches very briefly a definition of hyperplane without the use of an inner product, and states a theorem which is at least a partial generalization of Theorem I of the present paper. He attributes this work to Mazur without giving a specific reference; but it is possibly contained in [**10**], a paper which the present writer has not been able to see, and which was apparently overlooked by Bonnesen and Fenchel as it does not appear in their extensive bibliography.

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NEW ALGEBRAS OF LOGIC

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1. Introduction. In recent years, various new algebras of logic have been proposed which differ in certain ways from the classical Boolean algebra of logic. Three of the most interesting of these systems are the truth value logics of Lukasiewicz and Tarski [**1, 2**], Heyting's intuitionist logic [**3**], and the quantum mechanics logic of Garrett Birkhoff and von Neumann [**4**]. It must not be expected that these new logics will provide us with any new methods of drawing logical conclusions. On the contrary, they all require that we abandon one or more of the rules of classical logic, such as the law of excluded middle, or of double negation or the distributive law. However, these new logics do not agree with each other as to which particular Boolean rules are to be discarded. They are all interesting as purely abstract algebraic systems, apart from any logical interpretation. Since they are new, their algebraic properties have not yet been completely determined. It should be noted that they are algebras of elementary propositions, and are not adequate to represent all types of mathematical reasoning. For example, they contain no symbols meaning "for all x" or "there exists an x such that."

2. What is an algebra of logic? Just what is meant by an "algebra of logic"? This question may be answered by giving a description of Boolean algebra. Like the ordinary algebra of the real number system, a Boolean algebra consists of a set of elements, symbolized by the letters a, b, c, and so on, together with two operations or rules of combination called *addition* and *multiplication*, which assign to each pair of elements a and b of the algebra two further elements $a+b$ and ab called the *sum* and *product* of a and b. There is also an operation called *negation*, which assigns to each element a an element a' called the *negative* of a. These operations satisfy certain postulates given below, which are in part the same as postulates for ordinary algebra. Different algebras of logic, of course, will have different postulate systems.

The interpretation of the Boolean elements and operations, however, is entirely different from that of ordinary algebra. The elements a, b, c, and so on, are to be thought of as representing not numbers, but propositions or statements or sentences. The sum $a+b$ is to be thought of as standing for the proposition that either a or b is true, and the product ab is interpreted to mean the proposition that both a and b are true. The negation a' stands for the proposition that a is false. An algebra of logic may also have other operations, but in Boolean algebra all operations are definable in terms of addition and negation. All the algebras here discussed have the two special elements 0, standing for a proposition which is certainly false, and 1, representing a proposition which is certainly true.

3. The rules of Boolean algebra. We come now to the rules which the Boolean operations are assumed to obey. The notation $a=b$ used in stating these rules can be interpreted to mean that a and b are the same element, or it can be thought of as standing for an abstract equivalence relation meaning that each of the propositions a and b is a consequence of the other. Some of the Boolean rules are:

$$a+b=b+a \qquad ab=ba$$

$$(a+b)+c=a+(b+c) \qquad (ab)c=a(bc)$$

$$a+0=a \qquad a\cdot 1=a$$

$$a+1=1 \qquad a\cdot 0=0$$

These are the well known commutative and associative laws, and the special properties of the elements 0 and 1. The rule $a+1=1$ is the only one of these which fails in ordinary algebra. The rules above hold in all four systems to be discussed. The following Boolean rules fail in one or more of the new logics:

tautology: $a+a=a$ $aa=a$
distributive: $a(b+c)=ab+ac$ $a+bc=(a+b)(a+c)$
excluded middle: $a+a'=1$
contradiction: $aa'=0$
double negation: $(a')'=a$

To understand the significance of these rules for logic, the operations should be interpreted as above. For example, the law of excluded middle reads: For every proposition a, the statement that either a is true or a is false is certainly true.

These laws occur in pairs, in accordance with the principle of duality, which states that from any rule or provable formula another provable formula can be obtained by interchanging sum and product, and interchanging the elements 0 and 1. Sum and product have similar properties, being definable in terms of each other as follows: $ab\equiv(a'+b')'$, $a+b\equiv(a'b')'$. The rules above are not independent; the ones on the right, for example, all follow from those on the left.

There is another important interpretation of Boolean algebra in which the elements $a, b, c, \cdots$ stand for subsets of some "universal" set 1. In this interpretation $a+b$ means logical sum or union, and ab the logical product or common part of the sets a and b, and a' means the complement of a with respect to the set 1.

In Boolean logic, the operation of *implication* $a\supset b$ (read a implies b) is defined to be $a'+b$, that is, "either a is false or b is true." It should be noted that implication is not a *relation* between elements like equality which either holds or does not hold, but a rule of combination like addition, which produces from two elements of the algebra a third element. Many logicians do not like the definition of implication above and consider that it leads to properties that are paradoxical. Some of these properties of implication will be discussed later.

There exist Boolean algebras with $2, 4, 8, \cdots$ elements, the number of elements, if finite, being a power of 2. The number of elements may also be infinite. All Boolean algebras, however, have the same provable formulas (that is, algebraic identities) of which the rules above are examples. Since logicians are especially interested in provable formulas, they are apt to prefer the 2-element Boolean algebra as being the simplest. People interested in the relation between logic and the the theory of probability might prefer an infinite number of elements. It may be thought that the interpretation given to the law of excluded middle would rule out of consideration propositions which are neither certainly true nor certainly false, but merely probable. However, let a represent the merely probable proposition "It will rain tomorrow." Then $a+a'=1$ reads "The statement that either it will rain tomorrow or it will not, is certainly true," which seems plausible.

4. The Lukasiewicz-Tarski algebras of logic. In the Lukasiewicz-Tarski algebras of logic each element a has associated with it a real number from 0 to 1 called its truth value, $\tau(a)$. The truth value of a proposition may be interpreted to mean the probability that the proposition is true. In this logic, two propositions with the same truth value are considered to be logically equivalent. In other words, if $\tau(a)=\tau(b)$, then $a=b$. All logical operations are defined in terms of truth values alone.

There are Lukasiewicz-Tarski logics with 2 elements, 3 elements, and in general with any finite number of elements greater than 1. In the two-valued algebra (which is also a Boolean algebra) the truth values are 0 and 1. In the three-valued logic the truth values are 0, 1/2, and 1, and in the $n+1$-valued logic the truth values are $0, 1/n, 2/n, \cdots, 1$.

Lukasiewicz and Tarski take as fundamental operations *negation* and *implication* which they symbolize by Na and Cab. To make comparison easier we will use instead the notation a' and $a \supset b$, just as in Boolean algebra. Negation is defined in terms of truth values as follows:

$$\tau(a') \equiv 1 - \tau(a).$$

Thus if a has the truth value 1/2, its negative a' also has the truth value 1/2, and a is logically equivalent to its own negative. Such a proposition may be called doubtful, and its negative is also doubtful.

Implication has the following complicated definition in terms of truth values:

$$\tau(a \supset b) \equiv \min\,[1, \tau(a') + \tau(b)].$$

That is, if the sum of the truth values of a' and b is not more than 1, then this sum is taken as the truth value of $a \supset b$. For example, if $\tau(a)=.3$ and $\tau(b)=.6$, then $\tau(a \supset b)=1$, but $\tau(b \supset a)=.7$. In other words, a certainly implies b, but to say that b implies a is to make a statement whose probability is .7. Of any two propositions, the one of lesser truth value certainly implies the other.

In order to compare the Lukasiewicz-Tarski logics with other systems, it is necessary to define addition and multiplication operations in them. This can be done in different ways. Lukasiewicz and Tarski define addition as follows: $a \vee b \equiv (a \supset b) \supset b$. We shall call $a \vee b$ the *logical sum* of a and b. As in Boolean algebra, the corresponding multiplication operation is defined to be $a \wedge b \equiv (a' \vee b')'$. We shall call $a \wedge b$ the *logical product* of a and b. Logical sum and product satisfy all the laws given for Boolean sum and product except the law of excluded middle and the law of contradiction. It is easily seen that $a \vee b$ is equal to that one of the two elements a and b which has the greater truth value, while $a \wedge b$ is equal to the one of lesser truth value.

Logical sum cannot be taken as a fundamental operation, since implication cannot be defined in terms of it. For this reason it is algebraically convenient to define another addition operation as follows: $a+b \equiv a' \supset b$. We shall call $a+b$ the *arithmetic sum* of a and b. Its interpretation is, "if not a, then b," which is close to the meaning "a or b." In the usual way, the corresponding multiplication operation, called *arithmetic product*, is defined to be $ab \equiv (a'+b')'$. Arithmetic sum and product satisfy all the Boolean laws given above except the law of tautology and the distributive law. Both logical and arithmetic sum and product satisfy the "truth value equation":

$$\tau(a) + \tau(b) = \tau(a \vee b) + \tau(a \wedge b) = \tau(a + b) + \tau(ab),$$

which is familiar from the theory of probability.

Arithmetic sum may be used to replace implication as a fundamental operation, and all operations may be defined in terms of negation and arithmetic sum. The resulting algebra differs from other algebras of logic in that numerical coefficients and exponents occur. Since the law of tautology does not hold, $a+a$ and $a \cdot a \cdot a$ are in general different from a, and may be written $2a$ and a^3.

The two kinds of operations, logical and arithmetic, of the Lukasiewicz-Tarski algebras share between them all the properties of Boolean sum and product. As a consequence, most Boolean formulas have analogs in the Lukasiewicz-Tarski logic. In translating a Boolean formula into this new logic, however, the Boolean sum must sometimes be replaced by logical sum, and sometimes by arithmetic sum. Take for example the Boolean formula:

$$a(a \supset b) = ab.$$

This becomes in the Lukasiewicz-Tarski logic:

$$a(a \supset b) = a \wedge b;$$

that is, arithmetic product occurs on the left and logical on the right. Consider the Boolean formula: $a \supset (b \supset c) = (ab) \supset c$, which may be read "$a$ implies that b implies c" is the same as "a and b together imply c." This continues to hold when ab is taken to be the arithmetic product, but not when ab is replaced by the logical product $a \wedge b$. However, the paradoxical Boolean formula:

$$(a \supset b) + (b \supset a) = 1$$

holds with either logical or arithmetic sum. It is surprising that the formula:

$$[a \wedge (a \supset b)] \supset b = 1$$

does not hold in the Lukasiewicz-Tarski logics. Its interpretation would be: "It is certainly true that given a, and the fact that a implies b, then b follows." This seems reasonable enough, nevertheless it fails to hold, for example, when a has the truth value 2/3, and b has the truth value 1/3.

5. A postulate system for the Lukasiewicz-Tarski logics. All provable formulas of any Lukasiewicz-Tarski algebra hold also in the two-valued algebra (which is a Boolean algebra) since all the algebras contain the elements 0 and 1. Thus every provable Lukasiewicz-Tarski formula is also a provable Boolean formula. Unlike the case of Boolean algebra, however, different Lukasiewicz-Tarski algebras with different numbers of elements have different provable formulas. For example, the law of tautology $a+a=a$ holds in the two-valued logic but not in the three-valued logic. Similarly, the law $a+a+a=a+a$ holds in the three-valued but not in the four-valued logic and so on. In general the more elements there are, the fewer the provable formulas. This suggests the question of finding what provable formulas hold in all Lukasiewicz-Tarski algebras.

I have attempted to answer this question by constructing the following postulate system which I believe is new. It applies to all Lukasiewicz-Tarski logics. The postulates concern a class of elements $a, b, c, \cdots$, one of which is the special element 0, and an undefined operation called *implication*, which assigns to every pair of elements a and b of the set, distinct or not, a unique element $a \supset b$ of the set.

Definitions: $a \vee b \equiv (a \supset b) \supset b$, $a' \equiv a \supset 0$, $1 \equiv 0'$.

The postulates are, for every a, b, and c:

I. $a \supset (b \supset c) = b \supset (a \supset c)$.

II. $a \vee b = b \vee a$.

III. $a \vee 0 = a$.

IV. Either $a \supset b = 1$, or $b \supset a = 1$.

V. There are exactly $n+1$ elements.

I shall outline the proof that a system satisfying these postulates is a Lukasiewicz-Tarski algebra. It is first shown that either $a \vee b = a$, or $a \vee b = b$. The order relation $a < b$ (read a precedes b) is defined to mean $a \vee b = b$ and $a \neq b$. It is proved that this is a relation of simple or serial order, and that if $a < b$, then $b' < a'$. Arithmetic sum $a+b$ is defined to be $a' \supset b$, and is shown to be commutative and associative.

Let a_1 be the element which immediately follows 0 in the order relation above. Define a_2 to be a_1+a_1, a_3 to be $a_1+a_1+a_1$, and so on. Then it is proved that the elements $0=a_0, a_1, a_2, \cdots, a_n=1$ are all distinct and hence comprise all the $n+1$ elements of the algebra. These elements combine according to the rules: $a_r' = a_{n-r}$, and $a_r + a_s = a_{r+s}$, unless $r+s>n$, in which case $a_r+a_s=a_n$. But these are just the rules of combination of the truth values $0, 1/n, 2/n, 3/n, \cdots, 1$ of the $n+1$-valued Lukasiewicz-Tarski logic, when negation and arithmetic addition are taken as the fundamental operations. Hence the postulates define a Lukasiewicz-Tarski logic.

The provable formulas which hold in all Lukasiewicz-Tarski logics are clearly those formulas which follow from the first four of these postulates.

6. Heyting's algebra of logic. Heyting's algebra of logic is a formal postulational development of those rules of logic which the intuitionists of Brouwer's school recognize as valid. Although such a formal treatment of logic is contrary to the spirit of intuitionism, it serves the purpose of defending the intuitionists from the charge made by certain of their critics that a consistent logic which denies the law of excluded middle is impossible [**5**, **6**].

In Heyting's logic the operations of *addition*, *multiplication*, *negation*, and *implication* are taken as fundamental. Since the law of double negation does not hold, it happens that these four operations are independent; no one of them is definable in terms of the other three. For this reason the Heyting postulate system is quite complicated, and it will not be given in full here.

Heyting uses the notation $\neg a$, $a \vee b$, $a \wedge b$, for negation, sum, and product respectively. Instead of this, the Boolean notation will be used here. Heyting, in common with many other logicians, uses the symbol of assertion $\vdash a$, which allows him to dispense with the equality sign. In Heyting's notation $a=b$ would be written $\vdash (a \supset b) \wedge (b \supset a)$.

Heyting's sum, product, and negation satisfy all the Boolean laws listed above except the laws of double negation and excluded middle. In particular, the law of contradiction holds, and since contradiction and excluded middle are duals, it can be seen that the principle of duality must be abandoned.

Since implication is not defined, its properties do not follow from those of the other operations, but must be assumed by postulate. One of the properties of implication is:

$$(a' + b) \supset (a \supset b) = 1,$$

that is, if a is false or b is true, then a implies b. With Heyting as in Boolean algebra, a false proposition implies any proposition, and a true proposition is implied by any proposition.

Unlike Boolean algebra, proof by *reductio ad absurdum* is not valid. For example, in Heyting's logic

$$[a' \supset (bb')] = a'',$$

that is, if to deny a leads to the contradiction bb', that means that a is "not false," but it does not follow that a is true. Other important Heyting formulas are:

$$(a + b)' = a'b',$$

$$(ab)' = a \supset b',$$

$$(a \supset b)' = a''b',$$

that is, the denial of a sum is a product, the denial of a product is an implication, and the denial of an implication is a product.

Four important *Boolean* properties of implication are:

$$a \supset (bc) = (a \supset b)(a \supset c),$$
$$(a + b) \supset c = (a \supset c)(b \supset c),$$
$$(ab) \supset c = (a \supset c) + (b \supset c),$$
$$a \supset (b + c) = (a \supset b) + (a \supset c).$$

Of these, the first two hold in Heyting's logic, but the last two do not, which is just as well, since the last two are usually considered to be paradoxical. The third, for example, states that if two premises a and b imply a conclusion c jointly, then one of them implies the conclusion singly. It is interesting to note what becomes of these formulas in the Lukasiewicz-Tarski logics. If sum and product are taken to mean logical sum and product, all four formulas continue to hold, just as in Boolean algebra. If the sum and product are taken to be the Lukasiewicz-Tarski arithmetic sum and product, in no case does the equality sign hold, but in every case the left side of the formula implies the right side. Common sense tells us that with the last two formulas the reverse should be true, and the right sides should imply the left, and such is indeed the case in Heyting's logic.

Instead of the law of double negation, in Heyting's logic we have a law of triple negation, $a''' = a'$. As a consequence of this law, all negative propositions of the form a' do satisfy the law of double negation. In fact, the negative propositions a' of a Heyting algebra constitute a Boolean subalgebra of the Heyting algebra. In this Boolean subalgebra of negative propositions, product, implication, and negation are the fundamental operations, and addition must be redefined in terms of them. The Heyting sum cannot be used, for the Boolean law of excluded middle $a' + a'' = 1$ (a is false or a is not false) does not hold with the Heyting sum, even for negative propositions.

More generally, it has been shown [7] that no formula of the type $P + Q = 1$ is provable in Heyting's logic unless one of the formulas $P = 1$ or $Q = 1$ is provable. There are no provable alternatives in Heyting's system. This accounts for the absence of some of the paradoxical Boolean formulas, many of which are in the form of alternatives, such as $(a \supset b) + (b \supset a) = 1$, or $(a \supset b) + (a \supset b') = 1$.

The paradoxical Boolean formulas are not avoided entirely, however. It can be shown that if $F = 1$ is a provable Boolean formula, then $F'' = 1$ is a provable Heyting formula. Thus all Boolean formulas have analogs in Heyting's logic, but instead of being true, they may be merely not false. Examples are $(a + a')'' = 1$, and $[(a \supset b) + (b \supset a)]'' = 1$, that is the law of excluded middle is not false, and it is not false that of any two propositions, one of them implies the other.

Another interpretation of the Heyting negation a' is "a is absurd" or "a is impossible," instead of "a is false." Then a'' means "it is impossible that a is impossible" or "a is certainly possible" rather than "a is not false." With this interpretation of negation the law of double negation or of excluded middle would not be expected to hold.

7. The Birkhoff-von Neumann algebra of logic. Garrett Birkhoff and J. von Neumann have written a paper on the logic of quantum mechanics [**4**] in which they maintain that the rules of Boolean logic do not all apply to the propositions of quantum mechanics. The important part played by linearity and linear spaces in quantum mechanics suggested to them that the system of all linear subspaces of some linear space might be taken as a model for a quantum mechanics logic, just as the system of all subsets of a given set serves as a model for Boolean logic.

In particular, the set of all points, lines, planes, etc. of an n-dimensional projective space is perhaps the simplest model. The *product* of two linear subspaces is defined to be their logical product or intersection. The *sum* of two linear subspaces of the projective space is taken to be, not their logical sum, since that is usually not a linear subspace, but rather the smallest linear subspace containing them both. There is a wide choice of operations to serve as a model for *negation*. Any dual involution of the space which has the property that no two related elements intersect can be used. An example is a polarity in an imaginary quadric.

The formal properties which the operations of this concrete model obey can then be assumed as part of the postulates for a quantum mechanics logic. In the logical interpretation of this abstract system the elements will of course represent propositions, while sum, product, and negation have their usual meanings *or*, *and*, *not*.

It is found that the operations obey all the rules listed above for Boolean algebra except the distributive law. This law is replaced by a weaker law called the *modular identity*:

$$ab + ac = a(b + ac)$$

or, with three terms,

$$ab + ac + ad = a(b + ac + ad).$$

In other words, in taking out a common factor from the sum of several terms, it is removed from one term but must be left in the others.

To insure that the system is different from Boolean algebra, it is further assumed that the special elements 0 and 1 are the only elements having a unique inverse. *Inverse* is defined as follows: the element x is said to be an inverse of a if $a+x=1$ and $ax=0$. It can be seen that the negation a' is always an inverse of a, and in Boolean algebra it is the only inverse. In the projective geometry model, however, elements in general have more than one inverse, since there are many ways of selecting the negation operation.

Up to this point the postulates are satisfied by any n-dimensional projective space. To make the system categorical, a further assumption is made that there exists a truth value or probability function $\tau(a)$ which assigns to each element a of the algebra a real number from 0 to 1, such that

$$\tau(a) + \tau(b) = \tau(a + b) + \tau(ab),$$

$$\tau(0) = 0, \qquad \tau(1) = 1,$$

where τ takes on all values from 0 to 1. The truth value $\tau(a)$ is interpreted to mean the probability that the proposition a is true. Two elements with the same truth value are not necessarily identical. The interpretation of the system as an n-dimensional projective geometry is ruled out by the last assumption, since with that interpretation it can be shown that $\tau(a)$ could take on only a finite number of values, being determined by the dimensionality of the subspace a. With this last postulate the system becomes abstractly equivalent to the continuous geometry which was the subject of the Colloquium Lectures delivered by von Neumann at the Pennsylvania State College in 1937.

There is no implication *operation* in the Birkhoff-von Neumann logic, although there is a *relation* of implication. Just as in Boolean algebra, a implies b if $a+b=b$. The operation $a'+b$, which in Boolean algebra is defined to be implication, has few of the properties in the Birkhoff-von Neumann logic one would expect of an implication operation. For example, if $a'+b=1$, it does not follow that the implication relation $a+b=b$ holds.

The lack of an implication operation is not an important defect. It was one of Boole's objects in constructing an algebra of logic to express all logical ideas in terms of sum, product, and negation, avoiding operations like implication which are neither associative nor commutative. The Birkhoff-von Neumann logic is difficult to work with as an algebra, despite the fact that its postulates are very similar to those of Boolean algebra. Not only is the failure of the distributive law inconvenient, but there is no method of transposing terms from one side of an equation to the other. In the other three algebras, $a=b$ is equivalent to $(a \supset b)(b \supset a)=1$.

8. Some comparisons. In comparing the three new algebras, the following list indicates the systems in which various Boolean laws fail to hold. L stands for the Lukasiewicz-Tarski logical sum and product, A for the arithmetic sum and product, H for the Heyting system, and Q for the Birkhoff-von Neumann quantum mechanics logic.

Tautology	A
Distributive	A, Q
Excluded middle	L, H
Contradiction	L
Double negation	H

There is no single Boolean formula which we have yet discussed which all three new logics agree in rejecting. There exist such formulas, however. One of them is

$$ab + ab' = a.$$

In the Lukasiewicz-Tarski and Birkhoff-von Neumann logics an analogous formula holds, namely:

$$ab + a(ab)' = a.$$

In the Lukasiewicz-Tarski logics there is the further analogous formula:

$$a \wedge b + ab' = a.$$

In the Heyting logic the dual formula holds:

$$(a + b)(a + b') = a.$$

As for the paradoxical properties of the Boolean implication operation, it can be said that the Lukasiewicz-Tarski logic avoids few of them, while the Heyting logic avoids most of them at the expense of abandoning other properties not usually considered paradoxical. The Birkhoff-von Neumann logic avoids them by not having an implication operation.

In conclusion, it might be mentioned that if one wishes to introduce a truth value function into logic, as is done in the Lukasiewicz-Tarski and Birkhoff-von Neumann logics, in order to establish a connection between logic and the theory of probability, it is not necessary to abandon any of the Boolean rules. It is easy to introduce real-valued functions satisfying the truth value equation into many Boolean algebras, as is done in the theory of measure in point set theory.

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ON THE ABSOLUTE CONVERGENCE OF POLYNOMIAL SERIES

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1. Introduction. By a polynomial series we shall understand an expansion of the form

$$\sum_{n=0}^{\infty} a_n P_n(z),$$

where $\{P_n(z)\}$ is a given sequence of polynomials in the complex variable z and $\{a_n\}$ is a given sequence of complex numbers. With such a series are associated two point-sets C_0 and C_a in the complex z-plane. C_0 is the set of points z for which the series converges and C_a the set for which it converges absolutely. Either C_a or both C_0 and C_a may be vacuous in special cases.

In the present paper we shall assume that $P_n(z)$ is a polynomial of degree n having all its n roots real and we shall be concerned with the corresponding set C_a exclusively. In a number of cases the geometric character of C_a is fairly well known. The following table gives a list of some important cases.* The table gives the interior of C_a rather than C_a itself. The latter may differ from its interior by points on the boundary or, in the third case, also by a finite number of isolated points.

Polynomial	Interior of C_a
z^n	$\lvert z\rvert < r$
$(z-a_1)^n \cdots (z-a_k)^n$	$\lvert z-a_1\rvert \cdots \lvert z-a_k\rvert < r$
$\frac{1}{n!} z(z-1)\cdots(z-n+1)$	$x > x_a$
$\frac{1}{n!} e^z \frac{d^n}{dz^n}(z^n e^{-z})$	$-\eta < \mathfrak{I}(\sqrt{z}) < \eta$
$e^{z^2} \frac{d^n}{dz^n}(e^{-z^2})$	$-\eta < \mathfrak{I}(z) < \eta$
$\frac{1}{2^n n!} \frac{d^n}{dz^n}(z^2-1)^n$	$\lvert z-1\rvert + \lvert z+1\rvert < 2a$

The difference in the character of C_a in the various cases is evidently due to the difference in the distribution of the zeros of the polynomials in question. It would seem to be of some interest to study how C_a depends upon known features of this distribution. The present note is devoted to some elementary aspects of this question.

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* The first two cases of the table do not need any explanations. In the third case we are dealing with a binomial series also known as Newton's interpolation series. The remaining cases involve the polynomials of Laguerre, Hermite, and Legendre respectively. That a Legendre series converges in an ellipse with foci at ±1 is well known, but no proof seems to exist in the literature for the Laguerre and Hermite cases. Cf., however, G. N. Watson, The harmonic functions associated with the parabolic cylinder, Proceedings of the London Mathematical Society, ser. 2, vol. 8, 1910, pages 393–421, and vol. 17, 1918, pages 116–148. These papers contain asymptotic formulas for the Hermitian polynomials and a solution of the problem of expanding an analytic function in an Hermitian series.

In this paper we shall let $\mathfrak{I}(w)$ denote the *imaginary* part *of* w, $\mathfrak{R}(w)$ the *real* part of w.

2. Induced absolute convergence. The first feature one notices in studying polynomial series is that absolute convergence is contagious. Absolute convergence at one point induces absolute convergence in a set of points unless the first point is a zero of infinitely many of the polynomials in the expansion or a limit point of such zeros. The oldest and best known instance of this phenomenon is given by the classical

THEOREM 1. *If a power series* $\sum_0^\infty a_n z^n$ *is absolutely convergent at* $z=z_0(\neq 0)$, *then it is absolutely convergent in the circle* $|z| \leqq |z_0|$.

It is well known that this theorem is the best of its kind. Another familiar instance is given by

THEOREM 2. *Let* $P_n(z)$ *have real zeros for all* n. *Let** $z_0 \epsilon C_a$, $z_0=x_0+iy_0$, $y_0>0$. *Then* $z \epsilon C_a$ *if* $z=x_0+iy$ *and* $-y_0 \leqq y \leqq y_0$.

This follows from the fact that for a real a,

$$|x_0+iy-a|<|x_0+iy_0-a| \quad \text{for} \quad -y_0<y<y_0,$$

so that $|P_n(z)|<|P_n(z_0)|$ on the line segment in question.

This theorem is the best possible in the following sense. Consider the sets of absolute convergence of all polynomial series absolutely convergent at $z=z_0$. The cross section of all these sets is precisely the line segment $z=x_0+iy$, $-y_0 \leqq y \leqq y_0$, of the theorem. It is clear that the cross section contains this line segment; this is the assertion of Theorem 2. Now consider the following three polynomial series where $a_n=(n \log^2 n)^{-1}$:

$$\sum_{n=2}^{\infty} a_n(z/z_0)^n, \tag{1}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} a_n(z-x_0-1)(z-x_0-2)\cdots(z-x_0-n), \tag{2}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} a_n(z-x_0+1)(z-x_0+2)\cdots(z-x_0+n). \tag{3}$$

The corresponding C_a's are $|z| \leqq |z_0|$, $x \geqq x_0$, and $x \leqq x_0$ respectively. The cross section of these three sets is precisely the line segment in question so that the cross section of all sets C_a cannot be any larger set. This completes the proof of the assertion.

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* We recall that the notation $z \epsilon S$ is used to express that the point z is an element of the set S.

We get better results by restricting the position of the zeros. This will be done in the subsequent sections of the paper.

3. Positive zeros. In the present section the zeros of $P_n(z)$ will be supposed to be positive or rather non-negative for every n. Theorem 2 can then be replaced by

THEOREM 3. *Let* $P_n(z)$ *have real, non-negative zeros for all* n. *Let* $z_0 \epsilon C_a$, $z_0 = x_0 + iy_0$, $y_0 \neq 0$ *unless* $x_0 < 0$. *Then* $z \epsilon C_a$ *if* $|z| \leqq |z_0|$ *and* $x \geqq x_0$.

Proof. If $a \geqq 0$ and $z = x + iy$ is so chosen that $|z| \leqq |z_0|$, $x \geqq x_0$, then $|z-a| < |z_0 - a|$. Hence $|P_n(z)| < |P_n(z_0)|$, and the theorem follows.

This result is again the best of its kind in the sense that the cross section of the sets of absolute convergence of all polynomial series, absolutely convergent at $z = z_0$, and corresponding to polynomials with non-negative zeros, is precisely the region stated in the theorem. Indeed, it is the cross section of the C_a's corresponding to the series (1) and (2) above.*

As a consequence of Theorem 3 we shall prove the following well known theorem:

THEOREM 4. *The* C_a *of the general binomial series*

$$\sum_{n=1}^{\infty} a_n \frac{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)}{\lambda_1\lambda_2\cdots\lambda_n}$$

is a half-plane $x > x_a$ (*or* $\geqq x_a$). *Here* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lambda_n \rightarrow \infty$.

Remark. It follows from Theorem 6 below that x_a is always $-\infty$ when $\sum_{n=1}^{\infty} 1/\lambda_n < \infty$, *i.e.*, the half-plane becomes the whole plane.

Proof. We can write the series

$$\sum_{n=1}^{N} a_n \frac{(z-\lambda_1)\cdots(z-\lambda_n)}{\lambda_1\cdots\lambda_n} + \frac{(z-\lambda_1)\cdots(z-\lambda_N)}{\lambda_1\cdots\lambda_N} \sum_{n=N+1}^{\infty} a_n \frac{(z-\lambda_{N+1})\cdots(z-\lambda_n)}{\lambda_{N+1}\cdots\lambda_n}.$$

Suppose that the series is absolutely convergent for $z = z_0 \neq \lambda_n$ for every n, and choose N so large that $\lambda_{N+1} > k$ for a given positive k. Writing $z - \lambda_n = z - k - (\lambda_n - k)$ and applying Theorem 3 we see that the series is absolutely convergent in the region $x \geqq x_0$, $|z-k| \leqq |z_0 - k|$. But k can be taken as large as we please, and it follows that the series is absolutely convergent for $x > x_0$. Hence there exists a line $x = x_a$ to the right of which the series is absolutely convergent and to the left of which it can have this property only at those points λ_n such that $\lambda_n < x_0$.

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* If $x_0 < -1$, a slight modification is necessary since some of the zeros will then be negative. It is sufficient to suppress the corresponding factors in the terms.

Theorem 4 is only a special case of a more general theorem. The result is essentially due to the fact that the polynomials involved have only a finite number of zeros in each fixed interval (a, b). Let us denote the number of zeros of $P_n(z)$ in (a, b) by $N(P_n, a, b)$ and put $N(P, a, b)$ equal to the least upper bound of $N(P_n, a, b)$. We have then

THEOREM 5. *Let* $N(P, a, b) < \infty$ *for every finite interval* (a, b) *and let* $N(P, -\infty, 0)=0$. *Then if a point* z_0, *not on the positive real axis, belongs to* C_a, *the half-plane* $x > x_0$ *also belongs to* C_a.

Proof. Let $z_0=x_0+iy_0$, $z_1=x_1+iy_1$, $x_1>x_0$. Determine a real number ξ from the equation $|\xi-z_1| = |\xi-z_0|$. Since the case $\xi \leqq 0$ is covered by Theorem 3, we can suppose $\xi>0$. The number ξ always exists and is unique. Let us now factor $P_n(z)$ into two factors,

$$P_n(z) = P_{n,1}(z)P_{n,2}(z).$$

Here $P_{n,1}(z)=\Pi(z-a_{n,k})$ where $a_{n,k}$ runs over those zeros of $P_n(z)$ which are $\leqq \xi$. If there are no such zeros, $P_{n,1}(z)\equiv 1$. $P_{n,2}(z)$ contains the remaining factors. The degree of $P_{n,1}(z)$ is at most $N=N(P, 0, \xi)$. We have

$$|P_n(z_1)| = \left|\frac{P_{n,1}(z_1)}{P_{n,1}(z_0)}\right| \left|\frac{P_{n,2}(z_1)}{P_{n,2}(z_0)}\right| |P_n(z_0)|.$$

Here the quotients are products of fractions of the form

$$\left|\frac{z_1 - a_{n,k}}{z_0 - a_{n,k}}\right|.$$

The first quotient contains at most N such fractions in each of which $0 \leqq a_{n,k} \leqq \xi$. The maximum value of such a fraction is a certain quantity M, depending upon z_0 and z_1, the exact value of which is immaterial. In the second quotient all fractions are $\leqq 1$ by virtue of the determination of ξ. Hence

$$|P_n(z_1)| \leqq M^N |P_n(z_0)|.$$

It follows that the series is absolutely convergent at $z=z_1$, and the theorem is proved. That the theorem is the best of its kind follows from the example of series (2).

If we impose somewhat stronger restrictions on the zeros absolute convergence will be ensured in the whole plane.

THEOREM 6. *Let* $N(P, -\infty, 0)=0$ *and suppose that* $\sum_{k=1}^{n} 1/a_{n,k} \leqq M < \infty$ *for every* n. *Then if the polynomial series converges absolutely for a* z_0, *not on the positive real axis, it converges absolutely in the whole plane.*

Proof. In the following argument we shall need a couple of well known inequalities from the theory of the exponential function. We have $1+u \leqq e^u$ for all real u, and $1-u > e^{-2u}$ for $0<u<\frac{1}{2}$.

Let us define $P_n(z)=\prod_{k=1}^{n}[1-z/a_{n,k}]$. We begin by finding an upper limit of $|P_n(z)|$. We have clearly

$$|P_n(z)| \leqq \prod_{k=1}^{n} [1+|z|/a_{n,k}] < \exp\left\{|z|\sum_{k=1}^{n}(1/a_{n,k})\right\} \leqq \exp[M|z|],$$

where we have used the first inequality for e^u. Next we need a lower bound for $|P_n(z_0)|$. The inequality $\sum_1^n(1/a_{n,k}) \leqq M$ implies that $a_{n,k} \geqq k/M$, supposing $a_{n,k} \leqq a_{n,k+1}$. Since z_0 is not real positive number we can find a finite positive quantity $B(z_0)$ such that

$$\prod{}' |1 - z_0/a_{n,k}| > B(z_0),$$

where the product extends over those values of k for which $a_{n,k} \leqq 2|z_0|$. By virtue of the second inequality for e^u we see that the remaining factors, if any, satisfy the inequality

$$\begin{aligned}\prod{}''|1 - z_0/a_{n,k}| \geqq \prod{}''[1 - |z_0|/a_{n,k}] &> \exp\left[-2|z_0|\sum{}''(1/a_{n,k})\right] \\ &> \exp[-2M|z_0|].\end{aligned}$$

Hence

$$|P_n(z)| \leqq \frac{1}{B(z_0)} \exp[M(|z| + 2|z_0|)]|P_n(z_0)|.$$

This estimate suffices to prove the theorem.

We cannot replace the condition

$$\limsup_{n\to\infty} \sum_{k=1}^{n} 1/a_{n,k} < \infty$$

by the weaker assumption

$$\liminf_{n\to\infty} \sum_{k=1}^{n} 1/a_{n,k} < \infty.$$

In order to prove this it is sufficient to take

$$a_{2n+1} = 1, \qquad P_{2n+1}(z) = \binom{z-1}{2n+1},$$

$$a_{2n} = 1, \qquad P_{2n}(z) = \binom{z-1}{n}\binom{1-z}{n}.$$

This choice satisfies the weaker condition, but not the stronger one. The resulting series is absolutely convergent for $x>1$ and diverges for $x<0$.

We have seen that the domain of absolute convergence of a polynomial series is a half-plane if $N(P_n, a, b)$ is always bounded for finite intervals and $N(P_n, -\infty, 0)=0$. Here the first condition is merely sufficient and far from necessary. This is illustrated by the following rather special theorem:

THEOREM 7. *Let* $N(P_n, -\infty, 0) = 0$ *and*

$$\min (An^\alpha \omega, n) \leqq N(P_n, 0, \omega) \leqq \min (Bn^\alpha \omega, n),$$

where $0 < A < B$ *and* $0 < \alpha < 1$. *If a point* z_0, *not on the positive real axis, belongs to* C_a, *so does the half-plane* $\Re(z) > \Re(z_0)$.

Proof. Let $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$, $x_1 > x_0$. Let t be a real variable $\geqq 1$. An elementary calculation shows that

$$t^2 \left\{ \left| \frac{z_1 - t}{z_0 - t} \right| + \frac{1}{t}(x_1 - x_0) - 1 \right\} \equiv \eta$$

is bounded for $t \geqq 1$. The upper bound of $|\eta|$, M_0 say, is a function of z_1 and z_0, the exact value of which does not concern us. Hence

$$\left| \frac{z_1 - t}{z_0 - t} \right| = 1 - \frac{1}{t}(x_1 - x_0) + \frac{\eta}{t^2}, \qquad |\eta| \leqq M_0.$$

In the rest of the proof we use the same notation as in the proof of Theorem 5. The case $\xi \leqq 0$ being covered by Theorem 3, we suppose $\xi > 0$. For sufficiently large values of n we have

$$| P_{n,1}(z_1)/P_{n,1}(z_0) | \leqq \exp [n^\alpha B \xi \log M].$$

To every large n we can find a least integer m such that

$$a_{n,k} \geqq \rho \equiv \max [1, \xi, \tfrac{1}{2}(x_1 - x_0), 2M_0/(x_1 - x_0)], \qquad k \geqq m,$$

and for every such n, we have $m < Bn^\alpha \rho + 1$. For such values of k and n

$$\left| \frac{z_1 - a_{n,k}}{z_0 - a_{n,k}} \right| \leqq 1 - \tfrac{1}{2}(x_1 - x_0) \frac{1}{a_{n,k}} \cdot$$

On the other hand, if $\omega \leqq n^{1-\alpha}/A$ then

$$a_{n,k} \leqq \omega \quad \text{for} \quad k \leqq N(P_n, 0, \omega),$$

and

$$An^\alpha \omega \leqq N(P_n, 0, \omega).$$

Combining these inequalities we get

$$1/a_{n,k} \geqq An^\alpha / k,$$

and

$$\left| \frac{z_1 - a_{n,k}}{z_0 - a_{n,k}} \right| \leqq 1 - \frac{A}{2}(x_1 - x_0) n^\alpha \frac{1}{k}, \qquad m \leqq k \leqq n.$$

Hence

$$
\begin{aligned}
| P_{n,2}(z_1)/P_{n,2}(z_0) | &\leqq \exp\left[-\frac{A}{2}(x_1 - x_0)n^\alpha \log (n/m) \right] \\
&\leqq \exp\left\{ -\frac{A}{2}(x_1 - x_0)n^\alpha [(1-\alpha)\log n - \log B\rho] \right\},
\end{aligned}
$$

and finally

$$
| P_n(z_1) | \leqq \exp\left\{ -\frac{A}{2}(x_1 - x_0)n^\alpha [(1-\alpha)\log n - \log B\rho] + n^\alpha B\xi \log M \right\} \cdot | P_n(z_0) |.
$$

This estimate shows that $z_1 \epsilon C_a$ and completes the proof of the theorem.

The proof breaks down if $\alpha \geqq 1$ and the theorem is actually false for such an α. This is illustrated by the weighted binomial series

$$
\sum_{n=1}^{\infty} a_n \binom{n^\alpha z}{n}.
$$

If $\alpha > 1$, the series converges like a power series, *i.e.*, C_a is a circle $|z| < R$ or $|z| \leqq R$. If $\alpha = 1$, then C_a is bounded by a transcendental curve

$$
\Re[z \log z + (z - 1) \log (z - 1)] = \text{const}.
$$

These results are easily obtained from Stirling's formula in the theory of the Gamma function.

4. Symmetric zeros. The results derived for positive zeros are easily extended to symmetric zeros, *i.e.*, to polynomial series such that $P_n(-z) = \pm P_n(z)$. We have merely to separate the terms of even degree from those of odd degree and factor out a term z from the latter. The resulting series are polynomial series in z^2 to which the preceding analysis applies. It is sufficient to quote one single theorem as an example of the results so obtained. I take the analog of Theorem 5.

THEOREM 8. *Let the zeros of* $P_n(z)$ *be real and symmetric with respect to the origin and let* $N(P, a, b) < \infty$ *for every finite interval* (a, b). *Then if a non-real point* z_0 *belongs to* C_a, *the hyperbolic domain* $\Re(z^2) > \Re(z_0^2)$ *also belongs to* C_a.

EUCLID'S ALGORITHM FOR LARGE NUMBERS

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Euclid's algorithm which is so fundamental to the theory of regular continued fractions and elementary theory of numbers, is also of immense practical value in such well known problems as discovering the partial quotients in the regular continued fraction expansion of a given real number, and the solution of linear diophantine equations, to mention only two applications. In these problems, which occur frequently in experimental research, it is often necessary to carry the algorithm to a great many stages, as for example when one needs the greatest common divisor of two numbers each having, let us say, 30 digits, or when one wishes an extremely accurate rational approximation to a given irrational. In such cases, in which one would naturally use a computing machine, the algorithm involves numerous trivial operations with extremely large numbers. The purpose of this note is to show how more than 90% of these operations with large numbers may be eliminated. If one wishes historical evidence of the difficulties or rather the tedium of Euclid's algorithm in such cases, one may take the problem of expanding π in a regular continued fraction. In 1685 Wallis* computed the first 34 partial quotients of π. This calculation, made nearly a century before π was proved irrational, was verified as far as the 26th partial quotient by Lambert in 1770. But since then no one has extended the calculation of Lambert, and the fact that the 34th partial quotient given by Wallis should be 99 instead of unity has remained unnoticed until today.

1. Notation and general formulas. Let x_0 and x_1 be a pair of positive real numbers. Then Euclid's algorithm generates from x_0 and x_1 a set $\{x_\nu\}$ of real numbers and a set $\{q_\nu\}$ of integers by means of the equations

$$\begin{aligned} x_0 &= q_0x_1 + x_2, \\ x_1 &= q_1x_2 + x_3, \\ x_2 &= q_2x_3 + x_4, \\ &\cdots\cdots\cdots \\ x_\nu &= q_\nu x_{\nu+1} + x_{\nu+2}, \\ &\cdots\cdots\cdots, \end{aligned} \tag{1}$$

in which q_ν is the greatest integer not exceeding $x_\nu/x_{\nu+1}=\xi_\nu$. Eliminating $x_2, x_3, x_4, \cdots$ from (1) we obtain the regular continued fraction for ξ_0,

$$\xi_0 = \frac{x_0}{x_1} = q_0 + \frac{1}{q_1} \, \frac{1}{+ q_2 + \cdots} = [q_0, q_1, q_2, \cdots].$$

Similarly

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* John Wallis, A Treatise of Algebra, London, 1685, pp. 46–55.

$$\xi_\nu = \frac{x_\nu}{x_{\nu+1}} = [q_\nu, q_{\nu+1}, q_{\nu+2} \cdots]. \tag{2}$$

The nth convergent of ξ_0, namely

$$[q_0, q_1, q_2, \cdots, q_n],$$

is usually denoted by A_n/B_n, but to avoid the use of too many sub-subscripts in what follows we shall also write $A(n)$ and $B(n)$ for the numerator and denominator of the nth convergent of ξ_0. More generally, the nth convergent of ξ_ν,

$$[q_\nu, q_{\nu+1}, \cdots, q_{\nu+n}], \tag{3}$$

will be denoted by $A(n, \nu)/B(n, \nu)$ so that

$$A_n = A(n) = A(n, 0) \qquad \text{and} \qquad B_n = B(n) = B(n, 0).$$

The A's and B's satisfy the following recursion formulas

$$\begin{aligned} A(m, \nu) &= A(m-1, \nu)q_{m+\nu} + A(m-2, \nu), \\ B(m, \nu) &= B(m-1, \nu)q_{m+\nu} + B(m-2, \nu), \end{aligned} \tag{4}$$

with the initial conditions

$$\begin{aligned} A(-1, \nu) &= 1, \qquad & A(0, \nu) &= q_\nu, \\ B(-1, \nu) &= 0, \qquad & B(0, \nu) &= 1. \end{aligned} \tag{5}$$

If we eliminate $x_{\nu+2}, x_{\nu+3}, \cdots, x_{\nu+n-1}$ between the $(\nu+1)$st, $(\nu+2)$nd, $\cdots$ $(\nu+n-1)$st equations of the system (1) we obtain

$$x_{\nu+n} = (-1)^n\{B(n-2, \nu)x_\nu - A(n-2, \nu)x_{\nu+1}\}, \tag{6}$$

a formula which is readily proved by induction using (1), (4), and (5), and which lies at the root of our modification of the Euclid algorithm. The following formulas will also be of use and are easily established by induction from (4) and (5).

$$\begin{aligned} A_{\nu+n} &= A_\nu A(n-1, \nu+1) + A_{\nu-1}B(n-1, \nu+1), \\ B_{\nu+n} &= B_\nu A(n-1, \nu+1) + B_{\nu-1}B(n-1, \nu+1). \end{aligned} \tag{7}$$

Given the two numbers x_0 and x_1 having a very large number of significant figures, the application of Euclid's algorithm begins with the computation of the partial quotients q_ν $(\nu=0, 1, 2, \cdots)$. We note first that since q_0 is merely the greatest integer in x_0/x_1, crude approximations to x_0 and x_1 will in general be sufficient to determine q_0. More generally, if (x_0, x_1) are replaced by approximate values (y_0, y_1), then the first few partial quotients of x_0/x_1 and y_0/y_1 will be the same. If y_0 and y_1 have only a few significant figures, the first few partial quotients are easily obtained. Of course, if the Eulcid's algorithm for y_0/y_1 is carried too far, the q's will begin to disagree with those for x_0/x_1. There are two ways to find out how far the q's may be trusted. Perhaps the safest way is to choose a second pair (z_0, z_1) such that x_0/x_1 lies between y_0/y_1 and z_0/z_1. Then the partial quotients of y_0/y_1 and x_0/x_1 will agree at least as far as those of y_0/y_1 and z_0/z_1 agree. Since the whole calculation is based on the first few q's, the fact that they are obtained in two ways is a welcome check on the work, rather than a waste of effort. The other method of telling how far we may trust the q's will be discussed presently.

Having obtained the first few partial quotients, up to say q_{k_1}, we next compute the numbers

$$A(k_1, 0), \quad B(k_1, 0), \quad A(k_1 - 1, 0), \quad B(k_1 - 1, 0) \tag{8}$$

by the recurrence formulas (4) with $\nu = 0$. The relation

$$A(k_1, 0)B(k_1 - 1, 0) - A(k_1 - 1, 0)B(k_1, 0) = (-1)^{k_1 - 1}$$

affords an almost infallible check on the work.

Thus far the calculation has involved only small numbers. In fact we have not used x_0 and x_1, but merely approximations to them. By setting $\nu = 0$, and $n = k_1 + 1$ and $k_1 + 2$ in (6) we get

$$x_{k_1+1} = (-1)^{k_1+1}\{B(k_1 - 1, 0)x_0 - A(k_1 - 1, 0)x_1\}, \tag{9}$$

$$x_{k_1+2} = (-1)^{k_1+2}\{B(k_1, 0)x_0 - A(k_1, 0)x_1\}. \tag{10}$$

Here we encounter for the first time operations with really large numbers. To be quite certain that we are on the right track we may apply the following well known* theorem.

THEOREM A. *Let A_k/B_k, A_{k-1}/B_{k-1}, be two consecutive convergents to a number η. Then these fractions are consecutive convergents of ξ if and only if*

$$\left| \xi - \frac{A_k}{B_k} \right| < \frac{1}{B_k(B_k + B_{k-1})}.$$

Setting $\xi_0 = x_0/x_1$ and $k = k_1$ we obtain from (10) the condition

$$x_{k_1+2} < \frac{x_1}{B(k_1, 0) + B(k_1 - 1, 0)} \tag{11}$$

which is necessary for the correctness of the q's so far. We now replace x_{k_1+1} and x_{k_1+2} by approximations y_{k_1+1} and y_{k_1+2} and compute as before the first few partial quotients of y_{k_1+1}/y_{k_1+2}. By (2) these partial quotients will be

$$q_{k_1+1}, q_{k_1+2}, \cdots, q_{k_1+1+k_2}. \tag{12}$$

Using these q's we next compute the numbers

$$A(k_2, k_1 + 1), \quad B(k_2, k_1 + 1), \quad A(k_2 - 1, k_1 + 1), \quad B(k_2 - 1, k_1 + 1) \tag{13}$$

by (4) and check them by means of the relation

$$A(k_2, k_1 + 1)B(k_2 - 1, k_1 + 1) - A(k_2 - 1, k_1 + 1)B(k_2, k_1 + 1) = (-1)^{k_2 - 1}.$$

Using (6) with $\nu = k_1 + 1$ and $n = k_2 + 1$ and $k_2 + 2$ we get the equations

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* See for instance Lucas, Théorie des Nombres, p. 449.

$$x_{k_1+k_2+2} = (-1)^{k_2+1}\{B(k_2-1, k_1+1)x_{k_1+1} - A(k_2-1, k_1+1)k_{k_1+2}\}, \tag{14}$$

$$x_{k_1+k_2+3} = (-1)^{k_2}\{B(k_2, k_1+1)x_{k_1+1} - A(k_2, k_1+1)x_{k_1+2}\}, \tag{15}$$

which involve for the second time operations with large numbers. A test of the correctness of the set (12) of new q's is this time

$$x_{k_1+k_2+3} < \frac{x_{k_1+2}}{B(k_2, k_1+1) + B(k_2-1, k_1+1)}. \tag{16}$$

The process may now be continued using approximate values of $x_{k_1+k_2+2}$ and $x_{k_1+k_2+3}$ and obtaining a new set of q's, a new pair of A's and B's, and a new pair of x's. In this way the partial quotients may be extended as far as desirable.

In some problems the convergents A_n/B_n are of no interest. In these cases the above process is adequate. For example if we have a given number ξ_0 expressed let us say in decimals, and we wish to examine its partial quotients to see if they terminate, become periodic, or obey some law or other, we have only to choose $x_0 = \xi_0$, and $x_1 = 1$ and apply the above process. As a second example we may wish to find the greatest common divisor $\delta = (x_0, x_1)$ of two large integers x_0 and x_1. From (9) and (10) we see that any divisor common to x_0 and x_1 will divide x_{k_1+1} and x_{k_1+2} and conversely. Hence from (14) and (15) and all further similar equations we have

$$\delta = (x_0, x_1) = (x_{k_1+1}, x_{k_1+2}) = (x_{k_1+k_2+2}, x_{k_1+k_2+3}) = \cdots.$$

Since these x's decrease rapidly (by (11), (16), etc.) we soon come to a pair whose G.C.D. is easily found.

Of those problems in which the convergents are of importance a large majority require merely one convergent. This is the case for example when one wishes to get a rational approximation to a real number, or in the solution of linear diophantine equations, in which case the penultimate convergent of a rational number is needed. Sometimes one needs a sequence of convergents (or even intermediate convergents) beginning with A_n/B_n, the earlier convergents being of no use. This happens for example when one is looking for a rational approximation to a given real number which not only is sufficiently accurate but whose numerator or denominator has some further property.

In all these cases one may use formulas (7) to advantage as follows. We have already found the numbers (8), (13), etc. Substituting them in (7) with $\nu = k_1$ and first $n = k_2$ and then $n = k_2 + 1$, we get at once

$$\begin{aligned} A_{k_1+k_2} &= A_{k_1}A(k_2-1, k_1+1) + A_{k_1-1}B(k_2-1, k_1+1), \\ B_{k_1+k_2} &= B_{k_1}A(k_2-1, k_1+1) + B_{k_1-1}B(k_2-1, k_1+1), \\ A_{k_1+k_2+1} &= A_{k_1}A(k_2, k_1+1) + A_{k_1-1}B(k_2, k_1+1), \\ B_{k_1+k_2+1} &= B_{k_1}A(k_2, k_1+1) + B_{k_1-1}B(k_2, k_1+1). \end{aligned} \tag{17}$$

Thus we proceed from the numerators and denominators of one pair of consecutive convergents (8) to those of another isolated pair (17). Repeating the process as many times as is necessary one obtains a consecutive pair of convergents A_n/B_n, A_{n-1}/B_{n-1} for whatever value of n one may wish. The relation

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}$$

serves as a final check. One may also examine consecutive convergents (or intermediate convergents) in the neighborhood of A_n/B_n using (4) with $\nu=0$.

2. An application. To illustrate the foregoing method we consider the regular continued fraction for π. Taking a value of π correct to 100 decimal places* we shall first find the partial quotients

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, \cdots], \tag{18}$$

from which

$$\begin{aligned} A_{17} &= 2549491779, & B_{17} &= 811528438, \\ A_{18} &= 6167950454, & B_{18} &= 1963319607. \end{aligned} \tag{19}$$

Using now our 100 figure accuracy we find† (by (6) with $\nu=0$, $n=19, 20$)

$$\begin{aligned} x_{19} &= A_{17} - B_{17}\pi = 4.474\cdot 10^{-10}, \\ x_{20} &= -A_{18} + B_{18}\pi = 1.497\cdot 10^{-10}. \end{aligned} \tag{20}$$

As a check (11) gives

$$x_{20} < 1/(B_{18} + B_{17}) = 3.603\cdot 10^{-10}$$

which is in accord with (20).

Next we find that

$$\begin{aligned} \xi_{19} = x_{19}/x_{20} &= [q_{19}, q_{20}, \cdots, q_{32}, \cdots] \\ &= [2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, \cdots]. \end{aligned} \tag{21}$$

Hence

$$\begin{aligned} A(12, 19) &= 61245426, & B(12, 19) &= 20495141, \\ A(13, 19) &= 376962143, & B(13, 19) &= 126146437. \end{aligned} \tag{22}$$

Next we compute

$$\begin{aligned} x_{33} &= B(12, 19)x_{19} - A(12, 19)x_{20} = 1.185\cdot 10^{-18}, \\ x_{34} &= A(13, 19)x_{20} - B(13, 19)x_{19} = 1.188\cdot 10^{-20}. \end{aligned}$$

As a check (16) gives

$$x_{34} < x_{20}/(B(13, 19) + B(12, 19)) = 1.021\cdot 10^{-18}.$$

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* π=3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70680.

† The actual values of the x's have been suppressed to save space.

Next we find

$$\begin{aligned} \xi_{33} &= x_{33}/x_{34} = [q_{33}, q_{34}, \cdots, q_{50}, \cdots] \\ &= [99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, \cdots], \end{aligned} \tag{23}$$

from which

$$\begin{aligned} A(16, 33) &= 6185428223, \qquad & B(16, 33) &= 62033398, \\ A(17, 33) &= 7032882291, \qquad & B(17, 33) &= 70532479, \end{aligned} \tag{24}$$

and

$$\begin{aligned} x_{51} &= B(16, 33)x_{33} - A(16, 33)x_{34} = 1.249 \cdot 10^{-28}, \\ x_{52} &= A(17, 33)x_{34} - B(16, 33)x_{33} = 4.958 \cdot 10^{-29}. \end{aligned}$$

As a check

$$x_{52} < x_{34}/(B(17, 33) + B(16, 33)) = 8.965 \cdot 10^{-29}.$$

Next we find

$$\begin{aligned} \xi_{51} &= x_{51}/x_{52} = [q_{51}, q_{52}, \cdots, q_{76}, \cdots] \\ &= [2, 1, 1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, \cdots], \end{aligned} \tag{25}$$

and

$$\begin{aligned} A(24, 51) &= 2414289141, \qquad & B(24, 51) &= 958421828, \\ A(25, 51) &= 10210815077, \qquad & B(25, 51) &= 4053478055, \end{aligned} \tag{26}$$

from which

$$\begin{aligned} x_{77} &= B(24, 51)x_{51} - A(24, 51)x_{52} = 1.206 \cdot 10^{-38}, \\ x_{78} &= A(25, 51)x_{52} - B(25, 51)x_{51} = 7.099 \cdot 10^{-40}. \end{aligned}$$

As a check

$$x_{78} < x_{52}/(B(25, 51) + B(24, 51)) = 9.893 \cdot 10^{-39}.$$

Finally

$$\begin{aligned} \xi_{77} &= x_{77}/x_{78} = [q_{77}, q_{78}, \cdots, q_{90}, \cdots] \\ &= [16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, 24, \cdots], \end{aligned} \tag{27}$$

$$\begin{aligned} A(12, 77) &= 482872247, \qquad & B(12, 77) &= 28414566, \\ A(13\ 77) &= 11759887912, \qquad & B(13, 77) &= 692009353; \end{aligned} \tag{28}$$

hence

$$x_{92} = A(13, 77)x_{78} - B(13, 77)x_{77} = 7.328 \cdot 10^{-49},$$

whereas

$$x_{92} < x_{78}/(B(13, 77) + B(12, 77)) = 9.854 \cdot 10^{-49}.$$

Combining (18), (21), (23), (25), and (27) we have all the partial quotients of π from $q_0=3$ to $q_{90}=24$.

If we wish the 90th convergent A_{90}/B_{90} we proceed as follows. Setting $\nu=18$, $n=13$, 14 in (7) we have from (19) and (23)

$$\begin{aligned} A_{31} &= A_{18}A(12, 19) + A_{17}B(12, 19) = 430010946\ 591069243, \\ B_{31} &= B_{18}A(12, 19) + B_{17}B(12, 19) = 136876735\ 467187340, \\ A_{32} &= A_{18}A(13, 19) + A_{17}B(13, 19) = 2646693125\ 139304345, \\ B_{32} &= B_{18}A(13, 19) + B_{17}B(13, 19) = 842468587\ 426513207. \end{aligned} \tag{29}$$

As a check

$$A_{32}B_{31} - A_{31}B_{32} = -1.$$

Next setting $\nu=32$, $n=17$, 18 in (7) we obtain from (29) and (24),

$$\begin{aligned} A_{49} &= A_{32}A(16, 33) + A_{31}B(16, 33), \\ B_{49} &= B_{32}A(16, 33) + B_{31}B(16, 33), \\ A_{50} &= A_{32}A(17, 33) + A_{31}B(17, 33), \\ B_{50} &= B_{32}A(17, 33) + B_{31}B(17, 33). \end{aligned} \tag{30}$$

Next we set $\nu=50$, $n=25$, 26 in (7) and obtain from (30) and (26),

$$\begin{aligned} A_{75} &= A_{50}A(24, 51) + A_{49}B(24, 51), \\ B_{75} &= B_{50}A(24, 51) + B_{49}B(24, 51), \\ A_{76} &= A_{50}A(25, 51) + A_{49}B(25, 51), \\ B_{76} &= B_{50}A(25, 51) + B_{49}B(25, 51). \end{aligned} \tag{31}$$

Finally we set $\nu=76$, $n=14$ in (7) and obtain from (31) and (28),

$$\begin{aligned} A_{90} &= A_{76}A(13, 77) + A_{75}B(13, 77), \\ B_{90} &= B_{76}A(13, 77) + B_{75}B(13, 77). \end{aligned}$$

The actual values of A_{90} and B_{90} are

$$\begin{aligned} A_{90} &= 3062\ 43329\ 44969\ 82257\ 53216\ 23878\ 54374\ 05787\ 90366\ 50780, \\ B_{90} &= 974\ 80279\ 34167\ 85521\ 47430\ 34062\ 01616\ 85335\ 37806\ 95273. \end{aligned}$$

By Theorem A, $\pi - A_{90}/B_{90}$ should be less than $1.01 \cdot 10^{-96}$. By actual computation we find this difference to be less than $8 \cdot 10^{-97}$.

QUESTIONS, DISCUSSIONS, AND NOTES

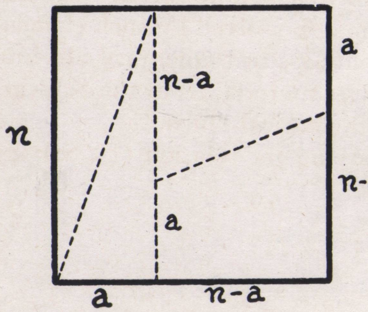
EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

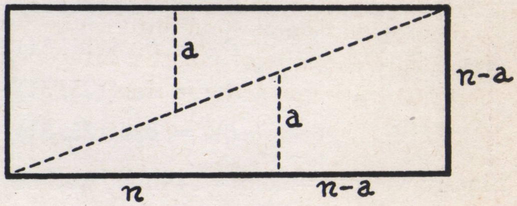
The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

LEWIS CARROLL AND A GEOMETRICAL PARADOX

WARREN WEAVER, Rockefeller Foundation

In a familiar geometrical paradox* a square of area $8\times 8=64$ square units is cut into four parts which may be refitted to form a rectangle of apparent area $5\times 13=65$ square units. In connection with editing unpublished mathematical material† left by Charles L. Dodgson (Lewis Carroll), the present author learned that Lewis Carroll generalized this paradox by characterizing the dimensions of all possible squares which may be cut in this same way and refitted into a rectangle with an apparent gain of one unit of area. Carroll's notes are incomplete and vague so that it is not possible accurately to reconstruct all details of his argument. Using the figure





he wrote the condition

$$(2n^2 - 3na + a^2) - n^2 = 1,$$

or

$$n^2 - 3na + a^2 - 1 = 0. \tag{1}$$

Assuming (or arguing in a way not stated) that a and n are the first two terms of a sequence of integers U_n any two successive terms of which produce an instance of the paradox, Carroll obtained (with details again uncertain) the difference equation

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* This paradox is described in W. W. R. Ball's Mathematical Recreations and Essays, 5th edition, London, Macmillan, 1911, p. 53. It apparently was first published in 1868 by Schlömilch. See Zeitschrift für Mathematik und Physik, vol. 13, p. 162.

† This material belongs to Mr. M. L. Parrish of Philadelphia, the owner of the finest Carroll collection in existence.

$$U_n + \Delta U_n - \Delta^2 U_n = 0,$$

for the determination of the sequence. Solving this equation by standard methods, he obtained the solution*

$$U_n = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{2^n\sqrt{5}},$$

and drew up the table

n	3	8	21	55	
a	1	3	8	21	etc.

The calculation is not dated, but Dodgson undoubtedly wrote this portion of the note book in question between 1890 and 1893. It seems clear that Dodgson was unaware of the similar generalization which was carried out much earlier by V. Schlegel.† A third (independent?) generalization was published by E. B. Escott.‡

The present note is written partly to call attention to the connection of the author of the immortal Alice in Wonderland with this well known geometrical puzzle; and partly to indicate that this problem may be solved by such elementary methods as to bring it within the range of a class in elementary algebra. Perhaps the argument, ballasted by a little heavier material from the Schlegel article, would be of some interest to junior mathematical clubs.

One seeks pairs of positive integers a, n satisfying the relation (1). Solving this quadratic for n, we obtain

$$n = \frac{3a \pm \sqrt{5a^2+4}}{2}. \tag{2}$$

Suppose one has a positive integral value of a which, when substituted into this formula, results in an integral value of n when one choice of sign is made. First one notes that the radicand must be a perfect square. If a be odd, then this radicand is odd and the positive square root is odd. But then both the sum and difference of the odd number $3a$ and the odd square root will be even, and not only one but both values of n are integral. The value, say n_-, which corresponds to the negative sign is non-negative and less than a, while the value n_+ is larger than a. The same end result holds if a be even. Thus we have three positive* integers n_-, a, and n_+ which we will call U_0, U_1, and U_2.

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* This formula was produced by J. P. M. Binet in Comptes Rendus de l'Académie des Sciences de Paris, tome 17, 1843, p. 563 in connection with a treatment of Fibonacci series. R. C. Archibald has pointed out the relationship between this geometrical puzzle and Fibonacci series. See this MONTHLY, vol. 25, 1918, p. 236.

† Zeitschrift für Mathematik und Physik, vol. 24, 1874, p. 123. This paper discusses various number theoretic properties of the sequence U_n, and includes the loss as well as the gain of one unit of area.

‡ Open Court Magazine, vol. 21, 1907, p. 502.

Moreovei (1) is symmetrical in n and a. Thus the equation

$$a = \frac{3n \pm \sqrt{5n^2 + 4}}{2} \tag{3}$$

must be satisfied by the pair of values $n = U_2$, $a = U_1$. But by exactly the argument above, if the integral value $n = U_2$ leads to a smaller value $a = U_1$, the choice of the plus sign will yield another integral value $a = U_3$ larger than U_2.

Thus it is clear that there exists an infinite sequence of integers $U_0, U_1, U_2 \cdots$ such that any successive pair U_r, U_{r+1}, constitute a pair of values of a and n which meet the condition of the problem. These numbers could be determined by successive use of (2). Beginning with any one known value of a and using the plus sign, one finds a value of n which is then used again in the formula as the next value of a, and so on.

But a somewhat simpler actual determination can be made, once the above facts are known. For we are now justified in writing

$$U_r^2 - 3U_rU_{r-1} + U_{r-1}^2 - 1 = 0,$$
$$U_{r+1}^2 - 3U_{r+1}U_r + U_r^2 - 1 = 0.$$

Subtracting and factoring, we obtain

$$(U_{r-1} - U_{r+1})(U_{r-1} + U_{r+1} - 3U_r) = 0,$$

so that

$$U_{r+1} = 3U_r - U_{r-1}. \tag{4}$$

One notices at once that $a = 0$ leads, in (2), to the integral value $n = 1$. With this beginning, as many terms as are desired are readily computed from the recurrence relation (4). The argument that one thus obtains all the possible pairs of integral solutions of (1) can well be left to the elementary algebra classes.

Editorial Note. In a letter to the Editor, Dr. Weaver gives a further interesting side light on the source material for the present article. He writes: "I have been engaged some little time in studying and editing the mathematical manuscript material which was left by C. L. Dodgson (Lewis Carroll). A large box of mathematical manuscripts all carefully catalogued as to subject, was willed by Lewis Carroll to one of his friends and subsequently sold to Mr. M. L. Parrish of Philadelphia." One of these manuscripts is the inspiration for the present paper. R. E. G.

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* The smallest may be zero.

COLLINEATION GROUPS WHICH ARE TRANSITIVE ON THE RATIONAL POINTS OF A CONIC

G. B. HUFF, Southern Methodist University

In a note in the MONTHLY, William Sell* showed that there exists a linear homogeneous transformation group generated by four involutions which would send any integral solution of $A^2+B^2=C^2$ into any other integral solution. The editor noted (*l. c.*) that this had interesting geometric implications when A, B, C are interpreted as homogeneous coördinates in the plane. In that case the involutions were harmonic perspectivities, each one defined by a point and its polar line with respect to the conic. From this point of view the generators were not independent and it was noted that three involutions were sufficient to generate the whole set of rational points on the conic from any one of them. The set, however, was not symmetric and the question was raised as to the possibility of the existence of a symmetric set of generators.

To examine this question we will show that the study of the ternary group leaving a proper conic invariant may be reduced to the study of a binary group. From this point of view it will be clear that many sets of generators may be devised. None of those shown are symmetric, and it is shown that the entire group is not generated if (a) the center of each involution is on the axes of the other two, or if (b) the centers of the involutions lie by pairs on tangent lines of the conic.

1. We denote any proper conic with rational coefficients by $(xx)=0$ and let (ax) represent the first polar of a point $a:(a_0, a_1, a_2)$ with respect to (xx). We seek to obtain a parametric representation of $(xx)=0$ and find the binary involution induced on the parameters by the harmonic perspectivity in a and its polar line $(ax)=0$.

If z is any point on $(xx)=0$ and if a is a point different from z, then the further intersection of $\lambda z+\mu a=0$ with $(xx)=0$ is given by $2\lambda(az)+\mu(aa)=0$. Thus that point, z', is determined by the equations:

$$z' = (aa)z - 2(az)a. \tag{1}$$

Equations (1) may be interpreted in another way. If a is not on the conic, it can be the center of an involution. The image z' of a conic point z is found by joining it to a and finding the further point of intersection. Thus the equations of the plane involution in a are:

$$x' = (aa)x - 2(ax)a. \tag{1'}$$

Let w, y be two distinct rational points of the conic different from z. Then $\alpha w+\beta y$, where α and β are rational gives all rational points of the line joining w, y. The line joining any one of these to z meets the conic in a rational point. A particular point on the conic is determined uniquely by α, β only after the point 1, 1 is chosen on the line $\overline{wy}$. This can be done conveniently by letting

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* William Sell: On integral solutions of $A^2+B^2=C^2$, this MONTHLY, vol. 43, 1936, pp. 481–3.

$(zx)=0$ meet $\overline{wy}$ where $\alpha=\beta$. This will be true if we write $a_{\alpha,\beta}\equiv\alpha(yz)w-\beta(wz)y$. Thus, $z_{\alpha,\beta}$, the point on the conic obtained by joining z to $a_{\alpha,\beta}$ is given by:

$$z_{\alpha,\beta} = (a_{\alpha,\beta}a_{\alpha,\beta})z - 2(a_{\alpha,\beta}z)a_{\alpha,\beta}. \tag{2}$$

Evaluating the symbols we have:

LEMMA I. *If w, y, z are three distinct rational points on the proper conic with rational coefficients $(xx)=0$, that conic has the parametric representation*

$$z_{\alpha,\beta} = (\alpha^2 - \alpha\beta)(yz)w + (\beta^2 - \alpha\beta)(wz)y + \alpha\beta(wy)z, \tag{3}$$

where w, y, z are the points $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively. As α, β run over all relatively prime integral values, this gives all the rational points on the conic.

The plane involution T_a with center at a and equations $(1')$ permutes the points of the conic and hence induces a transformation on the parameters α, β. To obtain the equations of this involution we note that it is completely defined by its fixed points which are the points in which the line $(ax)=0$ meets the conic. Substituting $z_{\alpha,\beta}$ in $(ax)=0$, we obtain

$$(\alpha^2 - \alpha\beta)(yz)(aw) + (\beta^2 - \alpha\beta)(wz)(ay) + \alpha\beta(wy)(az) = 0,$$

or

$$(yz)(aw)\alpha^2 + [(wy)(az) - (aw)(yz) - (ay)(wz)]\alpha\beta + (wz)(ay)\beta^2 = 0.$$

From the classical theory, we know that

$$t\colon \quad \begin{aligned} \alpha' &= -\, b\alpha - 2c\beta, \\ \beta' &= 2a\alpha + b\beta, \end{aligned}$$

is the involution whose fixed points are determined by

$$a\alpha^2 + b\alpha\beta + c\beta^2 = 0.$$

Hence T_a corresponds to the binary involution t_a:

$$t_a\colon \quad \begin{aligned} \alpha' &= -\, [(wy)(az) - (aw)(yz) - (ay)(wz)]\alpha - 2(wz)(ay)\beta, \\ \beta' &= 2(aw)(yz)\alpha + [(wy)(az) - (aw)(yz) - (ay)(wz)]\beta. \end{aligned} \tag{4}$$

If $a=\lambda(yz)w+\mu(wz)y+\nu(wy)z$, this takes the form:

$$t_a\colon \quad \begin{aligned} \alpha' &= -\, [-\, 2\nu(wz)(wy)(yz)]\alpha - 2(wz)[\lambda(yz)(wy) + \nu(wy)(yz)]\beta, \\ \beta' &= 2(yz)[\mu(wz)(wy) + \nu(wy)(wz)]\alpha + [-\, 2\nu(wz)(wy)(yz)]\beta. \end{aligned}$$

LEMMA II. *If $(xx)=0$ is given the parametric form* (3), *and if a is written*

$$a = \lambda(yz)w + \mu(wz)y + \nu(wy)z$$

then the binary involution corresponding to T_a is:

$$t_a\colon \quad \begin{aligned} \alpha' &= \nu\alpha - (\lambda + \nu)\beta, \\ \beta' &= (\mu + \nu)\alpha - \nu\beta, \end{aligned} \qquad \Delta = \begin{vmatrix} \nu & -\,(\lambda+\nu) \\ (\mu+\nu) & -\,\nu \end{vmatrix}. \tag{5}$$

The determinant $\Delta=\lambda\mu+\lambda\nu+\mu\nu$ cannot vanish since this is the condition that a be on $(xx)=0$.

2. We may now obtain a set of generators for the ternary group by devising a set of binary involutions which will generate all the pairs of integers α, β from any given pair. Consider the set of binary involutions:

$$(6) \qquad t_1\colon \begin{aligned} \alpha' &= \alpha - 2\beta, \\ \beta' &= -\beta; \end{aligned} \qquad t_2\colon \begin{aligned} \alpha' &= \alpha + \beta, \\ \beta' &= \alpha - \beta; \end{aligned} \qquad t_3\colon \begin{aligned} \alpha' &= \beta, \\ \beta' &= \alpha. \end{aligned}$$

The group generated by these contains $t_4=t_2t_3t_2$ which has the equations:

$$t_4\colon \begin{aligned} \alpha' &= -\alpha, \\ \beta' &= \beta. \end{aligned}$$

It is evident that (0,1), (1, 1), (1, −1), (1, 0) are conjugate under these elements. Moreover t_3 and t_4 can reduce any other point to one such that $\alpha>\beta>0$. Under t_1 this goes into a point such that $|\alpha'|<\alpha$, $|\beta'|=\beta$. This new point is either one of the first four or it may be reduced again. Thus the set of generators (6) will generate a group transitive on the rational points of the conic. The involution (5) takes the form of t_1, t_2, t_3 for $(\lambda, \mu, \nu)=(1, -1, 1), (-2, 0, 1), (1, -1, 0)$, respectively. Hence,

THEOREM I. *If* $(xx)=0$ *is any proper conic with rational coefficients, and if* w, y, z *are any rational points on the conic, then all the rational points of the conic are generated from any one by the plane involutions with centers at* $a_i=\lambda_i(yz)w+\mu_i(wz)y+\nu_i(wy)z$, $(i=1, 2, 3)$, *where* $\lambda_i, \mu_i, \nu_i=1, -1, 1; -2, 0, 1; 1, -1, 0$.

Other sets of generators may be devised in a similar manner. This particular set is used because it is simple and because the ternary generators reduce to those given by Sell for $(xx)=x_1^2+x_2^2-x_0^2$, and $w\colon(1, 0, 1)$, $y\colon(1, 0, -1)$, $z\colon(0, 1, 1)$.

3. The generators obtained in **2** are not symmetric. The center of T_1 is always on the axis of T_2 and the line joining the centers of T_2 and T_3 is tangent to the conic. Let us consider the cases where the two types of symmetry hold throughout.

When the center of each involution is on the axes of the other two, then any one is the product of the other two and the group generated is the "four group." Let a_1, a_2, a_3 be at the vertices of a self polar triangle and choose w, y as the points of tangency from a_1; choose z as either point of tangency from a_2. Then the values of λ, μ, ν determining a_1, a_2, a_3 may be computed and turn out to be:

$$a_1\colon\ (1, 1, -1); \quad a_2\colon\ (1, -1, 0); \quad a_3\colon\ (1, 1, 0).$$

Substituting these in (5) yields the binary involutions:

$$t_1\colon \begin{aligned} \alpha' &= -\alpha, \\ \beta' &= \beta; \end{aligned} \qquad t_2\colon \begin{aligned} \alpha' &= \beta, \\ \beta' &= \alpha; \end{aligned} \qquad t_3\colon \begin{aligned} \alpha' &= -\beta, \\ \beta' &= \alpha. \end{aligned}$$

Hence, we have the following:

THEOREM II. *If the centers of three plane involutions are at the vertices of a self-polar triangle of the conic* $(xx)=0$ *the group generated is the four group.*

4. Consider the case where the generating involutions are symmetric in that the centers lie in pairs on tangent lines of the conic. In this case the axes form an inscribed triangle with vertices at the points of tangency.

Choose w, y, z as the points of tangency. Then the points a_1, a_2, a_3 are given by the values of λ, μ, ν as follows:

$$a_1\colon\ (-1, 1, 1);\qquad a_2\colon\ (1, -1, 1);\qquad a_3\colon\ (1, 1, -1).$$

Substituting these in (5) yields:

$$t_1\colon \begin{array}{l}\alpha' = \alpha,\\ \beta' = 2\alpha - \beta;\end{array}\qquad t_2\colon \begin{array}{l}\alpha' = \alpha - 2\beta,\\ \beta' = -\beta;\end{array}\qquad t_3\colon \begin{array}{l}\alpha' = -\alpha,\\ \beta' = \beta.\end{array}$$

Evidently these can not generate the entire group since the parity of both α, β is invariant under each of these. By an argument similar to that in **2,** it may be shown that any characteristic other than (0, 1), (1, 1), (1, 0) may always be reduced to one of these. Since these are the points w, y, z (Lemma I), we may state the result:

THEOREM III. *If the centers of three involutions lie by pairs on tangent lines of the conic* $(xx)=0$, *the group generated is an infinite group which divides the rational points of the conic into three distinct conjugate sets. Each conjugate set contains one point of tangency and no point in one set is conjugate to a point in another set.*

The structure of this group is very interesting. For instance, any pair of the involutions generates an infinite dihedral group of the sort studied by Barber.*

5. The interesting point in this is that binary transformations are used to obtain results concerning plane collineations. The method may be used to answer other questions that may be raised in connection with the group of all collineations leaving a conic unaltered. The generating involutions given in **2** are unsymmetric exactly like those given by Sell and indeed are projective to them. By following the method of **2,** other sets may be devised which are not projective under rational transformations to these. No set, however, can exist which possesses either type of symmetry throughout.

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* S. F. Barber, Planar Cremona transformations, American Journal of Mathematics, vol. 56, 1933, p. 120.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Áctualités Scientifiques et Industrielles, Paris, Hermann, 1937. The following additional titles have been received. See this MONTHLY, vol. 42, 1935, pp. 40–41, and vol. 44, 1937, pages. 373–374, for earlier lists of titles in this series.

(Économique Rationnelle, exposés publiés sous la direction de G. et E. Guillaume, Nos. 1–5.)

504. *Méthode. L'économique élevée au rang d'une science exacte fondée axiomatiquement. Les conditions de prévisions pratiques.* 44 pages. 5f.

505. *L'économie pure. Le domaine des mécanismes élémentaires. Formation des prix et des salaries.* 76 pages. 8f.

506. *L'économie pure en interférence avec le domaine juridique. Mécanisme de la formation des bénéfices, des déficiences du pouvoir d'achat et des crises.* 55 pages. 6f.

507. *Le domaine économique-juridique national en interférence avec des mondes étrangers.* 59 pages. 6f.

508. *Modèles mathématiques des mondes économiques.* 136 pages. 15f.

Statistical Methods. Applied to Experiments in Agriculture and Biology. By G. W. Snedecor. Ames, Iowa, Collegiate Press, 1937. 13+341 pages. $3.75.

REVIEWS

Mathematical Snack Bar. By Norman Alliston. Cambridge, England, W. Heffer and Sons. Ltd., Distributed by Chemical Publishing Co. of N. Y. Inc., 1936. 7+155 pages. $3.00.

The book, as the author indicates in the preface, is a collection of notes and results obtained over a period of some twenty-five years of roving investigation. Modestly no claim is made either to great originality or to new results. The range of topics treated is wide including chapters devoted to the discussion of the triangle, quadrilateral, polygons in general, pyramids, ways of making a biquadratic expression square, residues, primes, Fermat problems, problems in Diophantine analysis, etc. Of these the ones dealing with triangles, quadrilaterals, and biquadratics I found the most interesting. One new point associated with the triangle is introduced and called the Tips Centre. The book is full of results and scattered bits of information; truly it is a mathematical snack bar.

But the most unusual thing about this volume is its striking language. If the mathematical investigator of today will consider the problem for just a moment he will realize that scientific language in general has now reached the point where it contains much more rigor than rhythm, more clarity than color, more verity than vigor, is much more propositional than persuasive, indeed is more logical than linguistical. Not so with the language of this book. Its structure is so curious and so condensed as to be very awkard so far as mathematical thought is concerned; occasionally, when read aloud, it sounds almost like a translation. At times it is quite non-understandable.

Although it will be somewhat difficult to use because of its language and also because of its utter lack of formality with regard to statements concerning the setting and background of problems and with regard to statements of definitions, theorems, proofs, etc., yet this little book may well prove to be an interesting and useful reference. It may be thought of as a companion volume to such books as Johnson's *Modern Geometry* and Carmichael's *Theory of Numbers.*

C. O. OAKLEY

Introduction to College Algebra. By William L. Hart. New York, D. C. Heath and Company, 1937. 5+246 pages. $1.84.

This text is intended for use as a three-hour semester course for college freshman who did not take advanced algebra in high school. It omits the following topics which are included in the author's *College Algebra*: mathematical induction, inequalities, complex numbers, permutations and combinations, probability, partial fractions, and infinite series. Determinants of the second and third order are introduced in connection with the solution of linear equations, and there is a paragraph on compound interest in the chapter on logarithms. The chapter on the theory of equations does not include Horner's Method or the algebraic solutions of the cubic and the quartic. The appendix contains a four place table of logarithms, a table of powers and roots, and a compound interest table. Answers are given for the odd exercises.

L. T. MOORE

Plane Trigonometry. By Mills, Atkin and Flagg. Chicago, Scott, Foresman and Company, 1937. 12+170 pages. $1.60.

The authors of this text have listed in the introduction an imposing array of seventeen features, among which may be mentioned: easy transition to new ideas, geometrical approach, emphasis on the use of natural functions, interesting applications, and historical notes.

The arrangement of the material differs from that of most texts in this subject as will be seen by noting the order of the nine chapter headings. They are: introduction, indirect measurement, ratios involving the sides of a right triangle, functions of any angle, solving right triangles, solving the general triangle, fundamental trigonometric relations, trigonometric functions of any angle in terms of the functions of an acute angle, logarithms and their use, solving triangles by the use of logarithms, complex numbers, and DeMoivre's theorem. Since the chapter on solving the general triangle precedes the chapters on trigonometric analysis, the formulas for the tangents of the half angles are derived from the two formulas for the area, *viz.*, $K=\sqrt{s(s-a)(s-b)(s-c)}$ and

$K=rs$. For the former of these the student is referred to any plane geometry.

There are some ingenious geometric proofs, but it is doubtful if the geometric proof of the Law of Tangents is as valuable to the student and at the same time as easy as the analytic derivation from the Law of Sines. The numerical problems are designed for use of the four-place tables included in the text in which the natural and the logarithmic functions of the angle are given at intervals of one degree.

At the beginning of Chapter V the authors very properly call attention to the two-fold nature of trigonometry, *i.e.*, the solution of triangles and trigonometric analysis, and they point out the importance of the latter phase of the subject for further study in mathematics. However, in the opinion of the reviewer, too much emphasis is placed on the former which may be called the arithmetic of trigonometry and too little on the latter, the algebra of trigonometry. The problems in identities and equations in both the direct functions and the inverse functions are too few and too easy to give the student necessary power for later work in calculus, the physical sciences, and engineering. This criticism is made in view of the fact that present-day instruction in elementary algebra includes the sine and tangent of acute angles, the use of tables, and the solution of right triangles by means of logarithms.

The book has an attractive appearance, the figures are especially well drawn, and the applications are interesting and wisely chosen, due importance being given to vector diagrams. The historical notes are very appropriate and the chapter on complex numbers deserves mention. It is unfortunate that many classes in trigonometry do not have sufficient time to include this topic.

J. I. TRACEY

Plane Trigonometry. By W. K. Morrill. New York, Farrar and Rinehart, 1938. 9+167 pages. $1.60.

This book was written primarily to provide a brief course enabling a student "after six or eight weeks of study, to proceed with analytic geometry and the calculus." However, starred sections are added for use in a more ambitious course. Thus most of the usual topics are included, except complex numbers.

The purpose of the author has been well carried out. Especially commendable are the exercises of which there are more than a thousand, with more difficult problems at the ends of most lists. The answers to the odd-numbered exercises are provided. There are no tables, but the explanations cover tables of different types. In this connection there is a misprint in the fourth line of the footnote on page 36.

The author uses the $\equiv$ symbol for identities. Example 3 on page 66 is open to criticism. Surely, in proving an identity, the left- and right-hand sides should not be stated to be identical until after it has been proved that each is equal to the same expression. The formulas for sine and cosine of a sum are proved only for acute angles, but the exercises suggest some extensions, and the formulas are stated to be true for all positive and negative angles. (Zero ?)

The book is attractively printed and bound and will be found to be well adapted for use in many courses.

J. W. CLAWSON

Plane Trigonometry. With Tables. By F. W. Sparks and P. K. Rees. New York, Prentice-Hall, 1937. 10+161+76 pages. $2.50.

The authors present the subject in a terse, direct style with clear but not elaborate explanations. The functions are defined at the start for angles of any magnitude in standard position. All of the usual topics are discussed, including inverse functions and complex numbers. The formulas for the cosine and sine of the sum of two angles are proved only for the case when each angle and the sum is acute; but it is stated that the validity for all angles may be established by similar arguments. The derivation of the dependent formulas is especially good. The reason why the double sign is omitted in the formula $\tan(\theta/2) = (1-\cos\theta)/\sin\theta$, is given on page 71, a matter often slurred over.

The reviewer does not always like the treatment of identities. In a worked-out example on page 21, the authors start with an identity to be proved, and, after several manipulations on both sides, end the treatment with the statement "Therefore $\sin^2 x = \sin^2 x$." Under inverse functions on pages 125 and 126, the same criticism applies.

The problems are adequate, except that many teachers would prefer to have more numerical problems with the parts of triangles given to four or five significant figures instead of two or three. Five-place logarithm tables and the answers to the even-numbered problems are provided. Unimportant misprints were discovered on pages 51 and 87; but the typography throughout is admirable.

J. W. CLAWSON

Analytic Geometry and Calculus. By M. Morris and O. E. Brown. New York, McGraw-Hill Book Company, 1937. 10+507 pages. $3.75.

The authors have adopted the point of view that a first course in analytical geometry should be a preparation for the calculus. To this end their choice of material is excellent. In the 104 pages of plane and 35 pages of solid analytical geometry there are only to be found those topics which the student will need in his work in calculus and elementary differential equations. There is nothing novel in the presentation, but the exposition is clear and the style attractive.

The first chapter of the part of the book dealing with the calculus is headed: Variables, Functions, Limits, and Continuity. The discussion of these topics is adequate and comprehensive. It would be difficult to improve on the method and illustrations used in developing the concept of limit. Detailed proofs are given of such theorems on limits as: *The limit of the product of two functions of a common variable is equal to the product of the limits*. In the reviewer's opinion the student has not, at the beginning of a first course in the calculus, the maturity to appreciate a careful proof of these theorems. But if at this stage proofs are given, those worked out by the authors of the book under review are as readable and convincing as one could hope for.

In dealing with derivatives the notation $f'(x)$ is used until the introduction of the differential. This procedure is none other than common sense dictates, but at this point far too few authors use common sense. There is a short and adequate section on infinitesimals concluding with a proof of Duhamel's theorem. This theorem involves two sets of infinitesimals, u_n and v_n, and an essential condition for the theorem to hold is that for a given $\epsilon>0$, $|v_n/u_n-1|<\epsilon$ for all n sufficiently great. The authors choose to state this in the equivalent form $|(u_n-v_n)/u_n|<\epsilon$ for all n sufficiently great. To the immature student the second form will seem artificial and mystifying. It is better to state the condition in the first form and change it over into the second as the proof progresses, especially since the theorem is later applied (p. 311) in terms of the first form.

Considered from the point of view of a first course, the introduction of the definite integral and the fundamental theorem of the integral calculus is satisfactory. The chapter on infinite series is especially well done. Mention should be made of an interesting section on curves, derived curves, and second derived curves, the relations among them, and their geometric significance. In addition to the material usually included in a first course, the book contains such topics as Rolle's Theorem, Theorem of the Mean, L' Hospital's Rule, Taylor's Theorem for a function of two variables, and some of the more delicate tests for the convergence of a series.

The atmosphere of the book is mature; but the style is appealing, and the whole set-up gives evidence of considerable class room seasoning and disillusionment, which makes it safe for a first course.

R. L. JEFFERY

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS FOR CLUBS

This list is a continuation from previous numbers of the MONTHLY; see December 1937 issue, and January and March of the present year.

17. *The Frozen Fountain.* By Claude F. Bragdon. New York, Knopf, 1932. 125 pages. $3.75. One of the best in a series by this author, dealing with the art of design in space and its application in architecture.
18. *A Primer of Higher Space.* By Claude F. Bragdon. New York, Knopf, 1923. 2nd revised edition. $2.00. Shows the use of projections of four dimensional regular solids in designs in architecture.
19. *The New Image.* By Claude F. Bragdon. New York, Knopf, 1928. 190 pages. $3.00.

20. *Brain Teasers*. By Philip J. Rulon. Boston, L. C. Page and Co., 1932. 250 pages. $1.50. Contains many mathematical amusement problems.
21. *Numbergrams*. By Norman F. Sparhawk. Boston, Van Press, 1932. 50 pages. $1.25. Arithmetic problems to be solved by substituting numbers for letters.
22. *Puzzles in Wood*. By Edwin M. Wyatt. Milwaukee, Bruce Publ. Co., 1928. 64 pages. $0.75. Directions for making puzzles, most of them mathematical, out of wood or cardboard.
23. *Historical Introduction to Mathematical Literature*. By G. A. Miller. New York, Macmillan, 1916. A guide book to the mathematical world before 1916. Includes the status of living mathematicians, mathematical societies and journals, international congresses, encyclopedias, a history of tables from the time of Ahmes, errors, and a bibliography.
24. *The Foundations of Science*. By Henri Poincaré. New York, The Science Press, 1921. Introduction by Josiah Royce. The chapters include: On the Nature of Mathematical Reasoning, Mathematical Magnitude, and Experience, The Non-euclidean Geometries, Space and Geometry, Relative Motion and Absolute Motion, Energy and Thermodynamics, The Calculus of Probabilities, Intuition and Logic in Mathematics, Mathematical Creation, and others.
25. *Science and the Human Temperament*. By Edwin Schroedinger. Norton, 1935. A collection of popular lectures, including an unusually illuminating and dignified discussion of the place of the scientist in a civilized society.
26. *Queen of the Sciences*. By E. T. Bell. Baltimore, Williams and Wilkins, 1931, $1.00. A somewhat conversational discussion of some of the foundations, frontiers, and fundamentals of modern mathematics.

CLUB REPORTS

1936–37

Junior Mathematical Club, University of Chicago

President, M. F. Smiley; Program Chairman, L. Green; Treasurer, H. Chatland; Social Chairmen, Miss M. B. Haberzetle and I. M. Niven. The following talks were given by graduate students: The hypothesis of the continuum; Generalized projective geometry; Positive ternary quadratic forms; Modular invariants; Order relations; Matrix approximations and generalized least square lines; Euclid's parallel postulate; Functions of matrices; Structure theorems of algebras; Hermitian series; A generalization of groups; Volterra's mathematical treatment of the struggle for existence. Besides these, the following lectures were also delivered: "A test for the convergence or divergence of a series of positive terms," by Professor T. Vijayaraghavan of the University of Dacca; "Some modern results in the theory of transcendental numbers," by Dr. G. C. Webber of the Armour Institute of Technology; "Some new mathematical problems in biophysics," by Professor N. Rashevsky of the University of Chicago. Besides the customary afternoon teas held for a half-hour prior to each meeting, the club also sponsored a number of bridge parties during the year.

Mathematics Club, Smith College

Secretary, Patricia A. Taggart. Topics discussed included: Origin of our numerals; Magic squares; Location of complex roots of polynomials; The Japanese bead-tray computator; A review of Stephen Leacock's article, *Humor thought in mathematics.* Four members of the Club were elected to Sigma Xi and of these, three were also elected to Phi Beta Kappa.

Pi Mu Epsilon, Lehigh University

President, C. H. Hoffman; Secretary-Treasurer, D. B. Wheeler; Director, Professor D. H. Lehmer. At the regular monthly meetings student members led the mathematical discussions which were frequently followed by talks by faculty members. The chapter sponsored a lecture by Professor F. W. Owens, Pennsylvania State College, on "Unusual coordinate systems" and a talk by Dr. W. A. Menisen of Bell Telephone Laboratories, inventor of the crystal clock, on "Precise time measurement." One evening the club members were guests at the home of Professor Tomlinson Fort. The year closed with a formal initiation banquet at which Dr. P. M. Palmer, Dean of Arts and Science College, was the principal speaker.

Mathematics Club, Oshkosh State Teachers College

President, A. Womaski; Vice-President, Dorothy Lindgren; Secretary, M. Batterman; Treassurer, K. Dornstreich; Faculty Advisers, Dr. M. M. Beenken, Dr. Irene Price. Monthly meetings were held with papers by student members and enlivened by mathematical recreations. Topics presented included: The International Congress; Egypt and the Egyptian pyramids; History of analytic geometry; Chinese contributions to mathematics; Finger printing; Educational value of mathematics; The San Francisco bridge; Mathematical pitfalls. The graduating members of the club were entertained at a mathematical party by the faculty advisers and the year closed with a successful picnic for all.

Mathematics Club, George Washington University

President, A. M. DuPrez, Jr.; Secretary-Treasurer, Marion M. Sandomire; Faculty Adviser, Dr. F. E. Johnston. Topics discussed included: Theoretical economics of consumer behavior; Some assumptions of geometry; Polygonal numbers; History of Fermat's last theorem; Application of mathematics in atmospheric optics; Star spectra; A new method of enumeration; Equi-long transformations and their geometry; Bessel's functions; Spheres related to the tetrahedron; Nomographs. Dr. Johnston gave an exhibition of rare books on mathematics together with some modern expositions of Galois's theory of groups and of non-euclidean geometry. The social events included a Christmas party and a spring picnic as guests of Dr. and Mrs. J. H. Taylor at Long Beach, Maryland.

Kappa Mu Epsilon, Alabama State Teachers College, Florence

President, B. Simmons; Vice-President, F. Riley; Secretaries, Martha Howland, Mary A. Richeson; Treasurer, Virginia Little; Faculty Adviser, Professor Orpha A. Culmer. This chapter in its second year's work concentrated on astronomy, sponsoring a lecture by Professor A. M. Harding, University of Arkansas. A picnic and an anniversary banquet were the chief social events.

Pi Mu Epsilon, University of Oklahoma

Director, Dr. J. C. Brixey; Vice-Director, J. Boler; Secretary, W. Kitchens; Treasurer, J. Long; Corresponding Secretary, Dr. Dora McFarland. The chapter sponsored the meetings of the University Mathematics Club. Topics included: Art and mathematics; Mathematical amusements; Numerology; Review of *Men of Mathematics*; Logarithms. The programs were designed to be of interest to the general public. At the initiation dinner talks were given on "Number systems" by Dr. S. B. Townes and on "Problems of a mathematics teacher" by L. Dwight. The chapter held its third annual mathematics prize contest which is restricted to students who have not entered their senior year.

Mathematics Club, Butler University

President, J. St. Helens; Vice-President, Mildred Rugenstein; Secretary, Margaret Stemp; Treasurer, J. Thurston. At the monthly meetings topics discussed included: Mathematical fallacies; Number systems; Mathematics in the United States; The calendar; Astronomy. Dr. B. C. Getchell gave a talk on "Personal experiences of the solar eclipse expedition in Maine." The Christmas party and a picnic were enjoyable social events.

Pi Mu Epsilon, University of Pennsylvania

President, D. L. Herr. This chapter divides its activities into three parts: First, monthly meetings at which brief papers are read by student members on some mathematical problems, either pure mathematics or some of its applications to other scientific fields; second, bi-monthly prize contests, open to the entire university body, which include solution of original problems chosen with reference to second and third year students; third, prizes awarded to the best student paper, presented each semester at a chapter meeting.

Pi Mu Epsilon, University of Missouri

This chapter, one of the oldest in the organization, offers an annual prize of $10.00 to the student, not yet a senior, who receives the highest mark in a special examination covering algebra, trigonometry, analytic geometry, and differential calculus. The meetings are devoted to subjects of pure and applied mathematics with one social meeting in addition to the annual initiation banquet.

Mathematics Club, University of Iowa

President, R. Little; Secretary, Margaret Kampmeier; Faculty Adviser, Professor N. B. Conkwright. Monthly meetings were held throughout the year and the following topics discussed: Conormal points on a parabola; A special one to one correspondence; Centro-surface of the ellipsoid; Some topics in the theory of numbers: Algebra and calculus of matrices; Mathematical recreations. The annual mathematics dinner for the senior class, graduate students, and faculty was the principal social event of the year.

Pi Mu Epsilon, Syracuse University

This is the original chapter of Pi Mu Epsilon. The record of their activities has been continuous since 1908. During the last year they have had various lectures by faculty members, a Christmas party, and the annual initiation banquet. They are laying plans for a more general program with students presenting most of the papers.

Mathematics Club, Connecticut College

Monthly meetings were held. Topics included: Building and loan associations; Theory of investment; Short cuts in mathematical computation; Curves; Works of Sir Isaac Newton. Marian Bills, Aetna Life Insurance Company, led a panel discussion on "Opportunities for women in life insurance work."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 327. *Proposed by Philip Franklin, Massachusetts Institute of Technology.*

It was shown by Fermat that, for any number k, it is possible to find k distinct right triangles, each with integral sides, having the same area. Find sets for $k=1, 2, 3, 4, 5$, having as common areas 6, 210, 840, $341880=2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 37$, and $37383746400 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 19$ respectively. Are these the smallest possible values?

E 328. *Proposed by C. A. Murray, West Texas State Teachers College.*

In the equation, $y = x^3 + px^2 + qx$, show how to determine all pairs of integral values of p and q, for which the equation $y = 0$ will have distinct integral roots, and the two bend points integral coördinates.

E 329. *Proposed by V. Thébault, Le Mans, France.*

How may we locate two points on the sides of a triangle so that the segment joining them may divide the area in a given ratio and have a maximum or minimum length?

E 330. *Proposed by Fred Discepoli, New York.*

Find a triangular number whose digit pattern in the decimal system is *aabb*, and determine whether or not the solution is unique.

E 331. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Determine a triangle whose six angle bisectors, three altitudes, five radii, and area are rational numbers.

E 332. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Prove that $1^p + 2^p - 3^p + 4^p - 5^p - 6^p + 7^p + 8^p - \cdots + (2^n - 1)^p = 0$ for every positive integer $p < n$, where the sign of a term of the form m^p is negative or positive according as m is or is not one of the 2^k integers following 2^{k+1}, where k is any integer.

SOLUTIONS

E 253 [1937, 49; 1937, 480]. *Proposed by V. Thébault, Le Mans, France.*

Find two positive integers differing by five, with the sum of their squares a perfect cube, and show that the solution is unique.

Comment by C. W. Trigg, Eagle Rock High School, Los Angeles.

On page 24, and in the subsequent theory, of his *Diophantine Analysis* Carmichael shows that the only solutions of $x^2 + y^2 = z^3$ are given by

$$x = m^3 + mn^2, \qquad y = m^2n + n^3, \qquad z = m^2 + n^2 \tag{1}$$

$$x = m^3 - 3mn^2, \qquad y = 3m^2n - n^3, \qquad z = m^2 + n^2. \tag{2}$$

If in (1) we set $x + 5 = y$, we get $m(m^2 + n^2) + 5 = n(m^2 + n^2)$, whence $(m^2 + n^2)(n - m) = 5$. Now $m \neq n$, so $n = 2$, $m = 1$, $x = 5$, $y = 10$, and $z = 5$. If we use $x - 5 = y$ in (2), we get $m(m^2 - 3n^2) - 5 = n(3m^2 - n^2)$, so that $(m + n)(m^2 + n^2 - 4mn) = 5$. Now, $m + n \neq 1$, so $m = 5 - n$. Hence $(5 - n)^2 + n^2 - 4n(5 - n) = 1$. That is, $6(n - 4)(n - 1) = 0$. Since $n < m$, $n = 1$, $m = 4$, $x = 52$, $y = 47$, and $z = 17$.

If we set $x + 5 = y$ in (2), the roots of the resulting equation in n are irrational. If we set $x - 5 = y$ in (2) we get, as in a previous case, $n = 1$, $m = 4$, $x = 52$, $y = 47$,

$z=17$. Therefore the two solutions published in the August-September 1937 number of this MONTHLY, page 480, are the only ones.

E 294 [1937, 539]. *Proposed by V. Thébault, Le Mans, France.*

Show that the difference between the cubes of two successive positive integers can never be either the double or the triple of a perfect square.

Solution by "Dumb," Dartmouth College.

Since $(x+1)^3-x^3=3x(x+1)+1$, it is divisible by neither 2 nor 3, and hence not the double or the triple of any integer. The restricting of the statement of the problem to the double or triple of a perfect square seems superfluous.

Editorial Note. In his solution, Joel Brenner proved that the difference of two consecutive cubes could not be divisible by multiples of 2, 3, 5, 11, or 17, and could not be 13 times a perfect square. Fred Discepoli proved that if the difference of two consecutive cubes were a prime times a perfect square, then the prime had to be one more than a multiple of 3.

Also solved by N. A. Brigham, J. H. Butchart, J. H. Cross, Daniel Finkel, M. J. Gottlieb, V. W. Graham, Joseph Milkman, F. C. W. Olson, J. Rosenbaum, E. P. Starke, C. W. Trigg, Z. W. Wilchinsky, and the proposer.

E 295 [1937, 539]. *Proposed by P. W. Lonergan, DeWitt Clinton High School, New York.*

In preparing a board for Chinese checkers, a conventional six-pointed star is formed by placing an equilateral triangle upon another of equal size, so that the trisection points of the sides of one coincide with those of the other. Each side of each "point" is divided into four parts, and the complete figure is cut into small equilateral triangles by drawing through each marked point, lines parallel to the other sides of the figure. Find the number of small equilateral triangles, and the total number of equilateral triangles of all sizes, in the complete figure.

Solution by W. E. Buker, Pittsburgh, Pa.

Orient the star so that, as on a map, two of the points would be due north and south. We count only the triangles with vertex north, which will be one-half the total. The following figures represent the number of such triangles, whose sides are progressively $1, 2, 3, \cdots, 12$ times the sides of the smallest triangle, with northward vertices:

$$96 + 75 + 58 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 394.$$

The total number of the smallest sized triangles is $2\cdot 96=192$, and of triangles of all sizes is $2\cdot 394=788$.

Also solved by C. W. Trigg.

E 296 [1937, 539]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

D, E, and F are the centers of the semicircles constructed on the sides BC, CA, and AB as diameters, and exterior to the triangle ABC. If d and e are the lengths of the common external tangents between the points of contact for the semicircles D and E, and D and F, construct triangle ABC, given d, e, and angle A.

Solution by V. W. Graham, The Harcourt Street High School, Dublin, Ireland.

Let the incircle touch BC, CA, AB at L, M, and N respectively, so that $AN=s-a$, etc., with the usual notation. If we know the radius of the incircle the solution is obvious, for drawing the incircle we obtain N and M, and since $AN\cdot NB=d^2$, this gives B, and similarly for C.

We are given $(s-a)(s-b)=d^2$ and $(s-a)(s-c)=e^2$, so that $a(s-a)=d^2+e^2$. Therefore $s(s-a)=(s-a)^2+a(s-a)=(s-a)^2+d^2+e^2=r^2\cot^2\frac{1}{2}A+d^2+e^2$.

Now $rs=[s(s-a)(s-b)(s-c)]^{1/2}$, so that

$$r = de/[s(s-a)]^{1/2} = de/[r^2\cot^2\tfrac{1}{2}A + d^2 + e^2]^{1/2}$$

$$= de\cdot\tan\tfrac{1}{2}A/[r^2 + (d^2+e^2)\tan^2\tfrac{1}{2}A]^{1/2}.$$

Now let

$$de\cdot\tan\tfrac{1}{2}A = q^2 \text{ and } (d^2+e)\tan^2\tfrac{1}{2}A = p^2,$$

so that p and q are known. Then if $r^2+p^2=t^2$, we have $rt=q^2$, and we see that our problem has reduced to another standard one (see Dilworth, *A new Sequel to Euclid*, page 84; the Ed. Company of Ireland) namely—Given the difference of the squares of two segments and their rectangle or product, to find their lengths.

Also solved by J. H. Butchart, Joseph Milkman, E. P. Starke, and the proposer.

E 297 [1937, 539]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

When the proper fraction, WPA/USA, is reduced to lowest terms, it becomes UV/VU. Here each letter represents a digit in the decimal system. Determine the original fraction and show that the solution is unique.

Solution by C. W. Trigg, Eagle Rock High School, Los Angeles.

In the first place, neither W, U, nor V is zero. V and U are not both even, and $U<V$. Furthermore, $U+V$ is not a multiple of 3.

In the second place, A is not zero, for if it were, we would have WP/US reducing to UV/VU, but $US<VU$.

In the third place, $WPA=k(UV)$ and $USA=k(VU)$, so k can not end in 1, 3, 7 or 9. If k is even, the above restrictions limit the values possible for UV to 16, 38, and 49. If k ends with 5, neither U nor V is 5, and the possible values of UV are 13, 17, 19, 37, and 79.

In the fourth place, the hundred's digit of USA is the same as the units digit of VU. This final restriction readily eliminates all of the remaining possibilities above with the unique exception of $185/365=37/73$.

Also solved by W. E. Buker, Fred Discepoli, William Douglas, V. W. Graham, C. E. Snoke, Jr., E. P. Starke, B. C. Zimmerman, and the proposer.

E 298 [1937, 539]. *Proposed by C. T. Holmes, Brunswick, Me.*

As a first step in making a string model of the hyperboloid, $x^2/a^2+y^2/b^2-z^2/c^2=1$, we drill holes through two pieces of board along the curve in which the surface is cut by the plane $z=Z$, that is, the ellipse

$$(1)\qquad x^2/(ka)^2+y^2/(kb)^2=1,\ z=Z,\quad \text{where}\quad k=\sqrt{(c^2+Z^2)}/c>1.$$

The rulings must cut the plane $z=0$ in the ellipse

$$(2)\qquad x^2/a^2+y^2/b^2=1,\ z=0,$$

which is similar to (1). Hence the projections of these rulings on the plane $z=0$ are chords of (1) which are tangent to (2) at their midpoints.

If through a point P_1 on the ellipse (1) we draw the chord P_1P_2 tangent to (2), then through P_2 draw the chord P_2P_3 tangent to (2), then through P_3 the chord P_3P_4 tangent to (2), and so on, what is the condition that we return ultimately to the point P_1, so that we shall not be under the tiresome necessity of drilling infinitely many holes?

Solution by W. B. Campbell, Drexel Institute of Technology.

If, in the coplanar ellipses (1) and (2) (with Z replaced by 0), we multiply the ordinates by a/b, we obtain the circles

$$(3)\qquad x^2+y^2=(ka)^2,$$

$$(4)\qquad x^2+y^2=a^2,$$

and any chords of (1) tangent to (2) become chords of (3) tangent to (4), all of equal length. Each chord then subtends a central angle $2\phi<\pi$, where $\sec\phi=k$, $\tan\phi=Z/c$. A continuous broken line of such chords will return exactly to its initial point on (3) if and only if $2\phi/2\pi$ is rational. If the ratio is expressed as the quotient p/q of two relatively prime integers, the qth chord will be the first so to return, and similarly for a set of chords connected with the given coplanar ellipses, for any choice of initial point on (1). [The same conclusion is reached by writing (1) and (2) in the parametric form $x=a\cos\theta$, etc., and finding the constant difference in eccentric angles of the two points in which (1) is cut by a tangent at any given point on (2).]

In the space model, alternate chords end on ellipse (1) in the upper plane $z=+Z$, and the string returns to the initial point thereon after q trips if q is even, or after $2q$ trips if q is odd, each trip contributing a ruled line to the desired locus. If $2n$ such lines are used, there will be n holes on ellipse (1), the eccentric angles of an adjacent pair of holes differing by $2\pi/n$. The distance between holes will vary approximately from $kb\cdot\sin(2\pi/n)$ to $ka\cdot\sin(2\pi/n)$. The minimum value of q is 3, for which $p=1$, $\phi=\pi/3$, $Z=c\sqrt{3}$, but this requires 6 trips. For $p/q=1/4$, $\phi=\pi/4$, and the fourth trip returns to P_0. Values of p/q meeting the requirements include (1/3), (1/4), (1/5), (2/5), (1/6), (1/7), (2/7), (3/7), (1/8), (3/8), (1/9), (2/9), (4/9), (1/10), (3/10), (1/11), (2/11), (3/11), (4/11), (5/11), (1/12), (5/12), (1/13), $\cdots$.

Also solved by E. P. Starke, Maud Willey, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3868. *Proposed by Arnold Dresden, Swarthmore College.*

Prove that, if $0 \leqq \alpha \leqq x_1 \leqq x_2$ and n is a positive integer, then

$$x_2^{1/n} - x_1^{1/n} \leqq (x_2 - \alpha)^{1/n} - (x_1 - \alpha)^{1/n}.$$

3869. *Proposed by Otto Dunkel, Washington University.*

Two perpendicular planes are tangent to a paraboloid and their intersection has a given direction. Determine the form of the surface generated by the intersection as the pair of planes varies. This supplements problem 3773 [1936, 111] solved 1938, page 124.

3870. *Proposed by A. Moessner, Nürnberg-N, Germany.*

The equations

$$A_1^3 + A_2^3 + A_3^3 + A_4^3 = B + C + D,$$

$$A_1^6 + A_2^6 + A_3^6 + A_4^6 = B^2 + C^2 + D^2,$$

$$A_1^3 + A_2^3 + A_3^3 = A_4^3,$$

are satisfied by $A_1=3$, $A_2=4$, $A_3=5$, $A_4=6$, $B=91$, $C=152$, $D=189$. What is the general solution in integers?

3871. *Proposed by V. Thébault, Le Mans, France.*

Let O, I, I_a, I_b, I_c be the centers of the circumcircle, inscribed circle, and the escribed circles of a given triangle ABC; and let Q_a, Q_b, Q_c, Q_a', Q_b', Q_c' be the intersections of the sides of the triangle with the interior and exterior bisectors of its angles. The parallels to the Euler line through the orthogonal projections on $Q_a'Q_b'Q_c'$, Q_bQ_c, Q_cQ_a, Q_aQ_b of an arbitrarily chosen point of that line meet respectively OI, OI_a, OI_b, OI_c in M, M_a, M_b, M_c. Show that the sum of the powers of O with respect to the circles with diameters IM, I_aM_a, I_bM_b, I_cM_c is zero.

SOLUTIONS

3730 [1935, 177]. *Proposed by William Chisholm, Quincy, Mass.*

There is said to be a short and simple method of finding one solution x of each of the equations

$$x^5 - (m + n)(5x^2 + 5x + 1) = (m + n)^2, \tag{1}$$

$$x^2 + m^2 + n^2 - 2(mn + mx + nx) = 8\sqrt{2mnx(x + m + n)}. \tag{2}$$

What is the method?

Solution of (2) *by V. W. Graham, The High School, Dublin, Ireland.*

The solution of (1) by F. Underwood was given in this MONTHLY, vol. 44, 1937, p. 111. For the solution of (2) set $x=y+m+n$, and there results

$$y^2 - 4mn = 8[2mn(y + m + n)(y + 2m + 2n)]^{1/2}. \tag{1}$$

Squaring each side and adding $16mny^2$ to each side of the resulting equation, we obtain

$$\begin{aligned} (y^2 + 4mn)^2 &= 16mn[3y + 4(m + n)]^2, \\ y^2 + 4mn &= \pm\, 4\sqrt{mn}[3y + 4(m + n)]. \end{aligned} \tag{2}$$

The solution of these two quadratic equations gives

$$\begin{aligned} y &= 6\sqrt{mn} \pm 4\sqrt[4]{mn}(\sqrt{m} + \sqrt{n}), \\ y &= -\,6\sqrt{mn} \pm 4i\sqrt[4]{mn}(\sqrt{m} - \sqrt{n}). \end{aligned} \tag{3}$$

The values of x are then easily obtained.

3778 [1936, 245]. *Proposed by L. J. Adams, High School, Beverly Hills, Calif.*

Solve

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\cdot$$

Solution by A. K. Waltz, Clarkson College, Potsdam, N. Y.

Set $x=e^u$, and the given equation becomes

$$y\left(\frac{dy}{du} - 1\right) = e^u. \tag{1}$$

Assume a solution of (1) in the form

$$y = \sum_{n=0}^{\infty} a_n u^n = \sum_{n=0}^{\infty} a_n(\log x)^n. \tag{2}$$

After substitution in (1) we have

$$\sum_{n=0}^{\infty} a_n u^n \left[\sum_{n=1}^{\infty} n a_n u^{n-1} - 1\right] = \sum_{n=0}^{\infty} u^n/n!.$$

Equating coefficients of the same powers of u on each side, we find

$$\begin{aligned} &(n + 1)\sum_{i=0}^{n/2} a_i a_{n+1-i} - a_n = \frac{1}{n!}, \quad n \text{ even;} \\ &(n + 1)\sum_{i=0}^{(n-1)/2} a_i a_{n+1-i} - a_n + \frac{n + 1}{2}\, a^2_{(n+1)/2} = \frac{1}{n!}\cdot \quad n \text{ odd.} \end{aligned} \tag{3}$$

The coefficients $a_1, a_2, \cdots$ in terms of a_0 may then be computed in succession. A general expression for a_n in terms of a_0 is not easily determined. The first three a's are

$$\begin{aligned} a_1 &= 1 + 1/a_0 = 1 + k, \qquad a_2 = k(1 - k - k^2)/2!, \\ a_3 &= [k - k^2(1 - k - k^2)(2 + 3k)]/3!, \qquad a_0 \neq 0. \end{aligned} \tag{4}$$

The equation (1) may be written

$$\frac{dy}{du} = 1 + \frac{e^u}{y} \cdot$$

The right side of this equation is analytic in u and y in the neighborhood of $u=0$ and $y=a_0\neq 0$. Hence there exists an integral $y(u)$ analytic in a neighborhood of $u=0$ which reduces to a_0 for $u=0$; and this integral may be obtained by the above method. (See Goursat-Hedrick's *Mathematical Analysis*, vol. 2, part II, page 45.)

The radius of convergence of the integral is not easily found, but it is greater than or equal to the radius given in (9) page 47 of the same reference.

3781 [1936, 245]. *Proposed by Otto Dunkel, Washington University.*

Evaluate

$$\sum_{k=1}^{n} \frac{\cos m(2k-1)\pi/n}{\sin^2 (2k-1)\pi/2n}$$

in terms of the integers m and n, $0 \leqq m \leqq n$; also the sum where the sine is in the numerator instead of the cosine.

Solution by the Proposer.

By the method of partial fractions we have

$$\frac{x^m}{x^n + 1} = -\frac{1}{n} \sum_{k=1}^{n} \frac{x_k^{m+1}}{x - x_k}, \qquad 0 \leqq m < n, \tag{1}$$

where m and n are positive integers and x_k is a root of $x^n+1=0$. If $m=n$, the equation (1) is true after adding unity to the right side. After taking the derivative of each member of (1), and the equation for $m=n$, and then setting $x=1$, we have

$$\frac{(2m-n)n}{4} = \sum_{k=1}^{n} \frac{x_k^{m+1}}{(1-x_k)^2}, \qquad 0 \leqq m \leqq n. \tag{2}$$

We now transform (2) by means of the following equations:

$$\begin{aligned} x_k = e^{(2k-1)\pi i/n}, \quad x_k^m + x_k^{-m} &= 2 \cos m(2k-1)\pi/n, \\ x_k^m - x_k^{-m} &= 2i \sin m(2k-1)\pi/n, \\ (1-x_k)^2 = -(1-x_k)(1-x_k^{-1})x_k &= -[2 - (x_k + x_k^{-1})]x_k \\ &= -2[1 - \cos (2k-1)\pi/n]x_k. \end{aligned} \tag{3}$$

We have first

$$\sum_{k=1}^{n} \frac{x_k^m}{1-\cos(2k-1)\pi/n} = \frac{n(n-2m)}{2}. \tag{4}$$

Since x_k^{-1} is also a root, we have another equation by replacing x_k in (4) by x_k^{-1}. The addition of these two equations gives

$$\begin{aligned} &\sum_{k=1}^{n} \frac{\cos m(2k-1)\pi/n}{1-\cos(2k-1)\pi/n} = \frac{n(n-2m)}{2}, \qquad 0 \leqq m \leqq n, \\ &\sum_{k=1}^{n} \frac{\cos m(2k-1)\pi/n}{\sin^2(2k-1)\pi/2n} = n(n-2m). \end{aligned} \tag{5}$$

We may combine the terms for k and $n-k+1$ and write

$$\sum_{k=1}^{[n/2]} \frac{\cos m(2k-1)\pi/n}{\sin^2(2k-1)\pi/2n} = \frac{n(n-2m)}{2} + \left\{\left[\frac{n}{2}\right] - \frac{n}{2}\right\}(-1)^m. \tag{6}$$

In a similar manner we get

$$\sum_{k=1}^{n} \frac{\sin m(2k-1)\pi/n}{1-\cos(2k-1)\pi/n} = 0. \tag{7}$$

The corresponding terms in (7) for k and $n-k+1$ cancel, and if there is a middle term it is zero.

We may also write

$$m = \frac{n}{2} - \frac{1}{n}\sum_{k=1}^{n} c_k^{(m)}, \qquad 0 = \sum_{k=1}^{n} s_k^{(m)}, \tag{8}$$

where $c_k^{(m)}$ is the summand in the first equation of (5), while $s_k^{(m)}$ is that in (7).

We shall now use the above results to prove the interpolation formula of Szegö mentioned in solution II of 3705 [1936, 246]. Let

$$\begin{aligned} f(\phi) &= A_0 + \sum_{m=1}^{n} (A_m \cos m\phi + B_m \sin m\phi), \\ g(\phi) &= \sum_{m=1}^{n} (A_m \sin m\phi - B_m \cos m\phi), \end{aligned} \tag{9}$$

where A_m, B_m are given constants. The theorem states that

$$g'(\phi) = \frac{n}{2} f(\phi) - \frac{1}{n}\sum_{k=1}^{n} \frac{f[\phi + (2k-1)\pi/n]}{1-\cos(2k-1)\pi/n}. \tag{10}$$

We have first by (8)

$$m \cos m\phi = \cos m\phi \left\{ \frac{n}{2} - \frac{1}{n} \sum_{k=1}^{n} c_k^{(m)} \right\} + \sin m\phi \left\{ \frac{1}{n} \sum_{k=1}^{n} s_k^{(m)} \right\}$$

$$\begin{aligned} &= \frac{n}{2} \cos m\phi - \frac{1}{n} \sum_{k=1}^{n} \frac{\cos m[\phi + (2k-1)\pi/n]}{1 - \cos (2k-1)\pi/n}, \\ m \sin m\phi &= \frac{n}{2} \sin m\phi - \frac{1}{n} \sum_{k=1}^{n} \frac{\sin m[\phi + (2k-1)\pi/n]}{1 - \cos (2k-1)\pi/n}, \\ 0 &= \frac{n}{2} \qquad\quad - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 - \cos (2k-1)\pi/n}. \end{aligned} \tag{11}$$

Then from

$$g'(\phi) = \sum_{m=1}^{n} (A_m m \cos m\phi + B_m m \sin m\phi)$$

and the three equations in (11) the desired result easily follows. We leave this final step to the reader since the equations are not easy to print.

3783 [1936, 309]. *Proposed by J. Barinaga, Madrid University.*

Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix} = 1!2!3! \cdots n! \left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1} \frac{1}{n}\binom{n}{n} \right].$$

I. *Solution by J. A. Greenwood, Duke University.*

Consider the determinant of the nth order differing from the one in the problem only in the exponent, which is $l+i-2$ for the ith row, where l is a positive integer and $i=2, 3, \cdots, n$. Subtract the first column from each of the following. This results in a determinant of order $n-1$ which may be expanded in the form

$$\sum_{r=1}^{n} (-1)^{r+1} \frac{(n!)^l}{r^l} V(1, 2, \cdots, r-1, r+1, \cdots, n), \tag{1}$$

where V is the Vandermonde determinant

$$\begin{aligned} &V(1, 2, \cdots, r-1, r+1, \cdots, n) \\ &= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & r-1 & r+1 & \cdots & n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^{n-2} & \cdots & (r-1)^{n-2} & (r+1)^{n-2} & \cdots & n^{n-2} \end{vmatrix} \\ &= \frac{(n-1)!(n-2)! \cdots 2!1!}{(r-1)!(n-r)!}. \end{aligned}$$

For $r=1$ we have $V(2, 3, \cdots, n)$. The determinant considered then reduces to

$$1!2! \cdots (n-1)!(n!)^{l-1} \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \frac{1}{r^{l-1}} \cdot$$

For $l=2$ we obtain the special case of the problem.

II. *Solution by Morgan Ward, California Institute of Technology.*

Let $F(x)$ denote the polynomial which is obtained from the determinant in the problem by replacing its first column by $1, x^2, x^3, \cdots, x^n$. The given determinant is then $F(1)$. We have first

$$F(0) = 1!2! \cdots n!n \tag{1}$$

by reductions similar to those used for Vandermonde determinants. Since $F(x)$ vanishes for $x=2, 3, \cdots, n$, we have

$$F(x) = A(x-2)(x-3) \cdots (x-n)(x-B), \tag{2}$$

where the constants A and B will now be determined. Taking the logarithmic derivative of the two members of (2), then setting $x=0$, and observing that $F'(0)=0$, we have

$$\frac{F'(0)}{F(0)} = -\sum_{i=2}^{n} \frac{1}{i} - \frac{1}{B} = 0. \tag{3}$$

Set $x=0$ in (2) and we have in turn

$$AB = (-1)^n 1!2! \cdots (n-1)!n,$$

$$A = (-1)^{n+1} 1!2! \cdots (n-1)!n \sum_{i=2}^{n} \frac{1}{i},$$

$$F(x) = (-1)^{n+1} 1!2! \cdots (n-1)!n(x-2)(x-3) \cdots (x-n) \left[x \sum_{i=2}^{n} \frac{1}{i} + 1 \right],$$

$$F(1) = 1!2! \cdots (n-1)!n! \sum_{i=1}^{n} \frac{1}{i} \cdot$$

Since

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \frac{1}{i},$$

this completes the proof.

Solved also by Frank Ayres, Jr., and J. F. Locke.

Editorial Note. The proposer stated that the equality in the problem is a simple consequence of a formula given by L. Theisinger in the *Monatshefte für Mathematik und Physik*, vol. 26, 1915, p. 134. It will be observed that the second as well as the first solution generalizes the problem. If we set $x=n+1$ in $F(x)$ in the second solution, and make slight simplifications on each side, we obtain the evaluation of a determinant related to the one of the problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The National Council of Teachers of Mathematics will have its fourth summer meeting with the N. E. A. in New York City on June 27, 28, 29. The general theme of the three day session will be "Mathematics that Functions." The general plan is as follows:

(1) "Progressive" Arithmetic Teaching, June 27, 2 p.m.
(2) "The Forgotten Pupil," a joint meeting with the Department of Secondary Education, June 27, 3 p.m.
(3) Discussion Luncheon, June 28, 12:30 p.m.
(4) "Progressive" High School Mathematics, June 28, 2:30 p.m.
(5) Mathematics as it Functions in Business and Industry, June 29, 2 p.m.

Details including speakers, places of meetings, reservations for luncheon will appear in the May issue of the *Mathematics Teacher*.

The symposium on the algebra of geometry and related subjects at the University of Notre Dame, announced in the January number of this MONTHLY, was held February 11 and 12, in five sessions. At the first session, on Friday morning, the following papers were presented: Boolean algebra and topology, by Professor M. H. Stone; Partially-ordered function spaces, by Mr. Garrett Birkhoff. At the session on Friday afternoon the following papers were presented: On structures and their applications, by Professor Oystein Ore; Applications of the lattice theory to field structure, by Dr. Saunders MacLane; Some problems concerning groups, by Dr. S. K. Senior. On Friday evening Professor E. V. Huntington gave a general lecture on the method of postulates, entitled: The duplicity of logic. On Saturday morning the following papers were presented: Continuous geometry, by Professor John von Neumann; Alt's algebra of affine geometry and related problems, by Professor Karl Menger; A theorem on the algebra of geometry, by Dr. A. N. Milgram. At the last session, on Saturday afternoon, the following papers were presented: An application of the theory of division algebras to geometry, by Professor A. A. Albert; An algebraic topic with geometric aspects, by Professor Emil Artin.

The Walker Ames Professorship of Mathematics represents an attempt on the part of the University of Washington to take advantage of the fine summer climate of Seattle to establish the latter as a center of mathematical activity in the northwest. Professor Weyl of the Institute for Advanced Study follows Professor Veblen, who initiated the Ames Professorship in the summer of 1937.

Assistant Professor R. G. Archibald of Columbia University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor A. E. Brandt of Iowa State College is on leave of absence this year. He is serving as senior statistician at the United States Bureau of Soils, Department of Agriculture.

Assistant Professor A. B. Brown of Columbia University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor S. S. Cairns of Lehigh University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor A. T. Craig of the University of Iowa has been promoted to an associate professorship.

Professor Peter Field of the University of Michigan has leave of absence for the second semester of 1937–38.

Associate professor C. A. Garabedian of Wheaton College, Massachusetts, has been promoted to a professorship.

Dr. Evan Johnson of the Uniontown Extension Center of the Pennsylvania State College has been promoted to an assistant professorship.

Professor Karl Menger of the University of Notre Dame has been made chairman of the department of mathematics.

Professors R. E. Moritz and G. I. Gavett of the mathematics department of the University of Washington will retire to half time teaching beginning with the fall quarter of the academic year 1938–39. Professor Moritz has served from 1904 to 1936 as executive secretary of the mathematics department. Professor Gavett has been a member of the department from 1907 to the present time.

Dr. D. C. Murdoch of the University of Toronto has been awarded a Sterling Fellowship at Yale University.

Dr. E. R. Ott has been promoted to an assistant professorship at the University of Buffalo.

Dr. E. L. Post of the College of the City of New York has been promoted to an assistant professorship.

Professor E. L. Rees of the University of Kentucky is on leave of absence the second semester 1937–38 and is traveling in South America.

Dr. W. O. Rogers of Pennsylvania State College has been promoted to an assistant professorship.

Professor J. A. Shohat of the University of Pennsylvania has been elected Fellow of the American Institute of Statistics.

Mrs. O. H. Stecker of the Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor J. M. West of the Pennsylvania State College has been promoted to an associate professorship.

Dr. R. B. Allen, professor emeritus of mathematics at Kenyon College, died March 4, 1938, at the age of sixty-six. He had taught at Kenyon College since 1906, retiring in June 1937. He was a charter member of the Mathematical Association.

Professor J. B. Meyer of the State Teachers College, Valley City, North Dakota, died on February 25, 1938. He had been a member of the Association since 1920.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1938.

Catholic University of America. In addition to the usual elementary courses the following advanced courses will be offered: By Dr. E. J. Finan: Theory of equations; Theory of groups. By Professor J. N. Rice: Analytic geometry of three dimensions; Calculus of observations. By Professor O. J. Ramler: College geometry; Differential equations; Synthetic projective geometry.

University of California at Los Angeles. In addition to the usual elementary courses the following advanced courses will be offered: By Dr. A. L. Foster: Advanced calculus. By Professor P. H. Daus: Elementary mathematics from an advanced standpoint. By Professor W. M. Whyburn: Calculus of variations. By Vice-President E. R. Hedrick: The teaching of mathematics.

University of Chicago. First term, June 17 to July 22; second term, July 23 to August 28. During the Summer Quarter of 1938 the department of mathematics at the University of Chicago will offer a well balanced program, but will place especial emphasis on algebra and the theory of numbers. A conference on algebra is to be held June 28 to July 1 at which a number of eminent mathematicians will present papers. Courses emphasizing some phase of algebra are: A seminar on topics in algebra, by Professor A. A. Albert and his associates; Modern higher algebra, by Professor Albert; Theory of algebraic numbers, by Dr. D. M. Dribin; Algebraic functions, by Professor G. A. Bliss; Continuous groups, by Dr. Nathan Jacobson; and Elementary theory of numbers, by Professor L. E. Dickson. In addition to the above courses and those in calculus, differential equations, and solid analytical geometry, the following courses will be given: By Professor W. T. Reid: Introduction to the theory of functions; Integral equations. By Professor M. R. Hestenes: Theory of functions of real variables. By Professor E. P. Lane: Metric differential geometry; Projective differential geometry.

University of Colorado. First term, June 20 to July 23; second term, July 25 to August 27. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor A. J. Kempner: Advanced calculus (both terms); Theory of numbers (both terms); Mathematical recreations (first term). By Professor C. A. Hutchinson: Differential geometry (both terms). By Professor Claribel Kendall: Theory of equations (second term).

Columbia University. The graduate courses in mathematics offered in the summer session of 1938 are as follows: By Professor Kasner: Survey of modern mathematics; Curves and families of curves. By Professor Koopman: Differential equations. By Professor P. A. Smith: Introduction to higher algebra. By Professor Fite: Theory of functions of a complex variable.

Duke University. First term, June 13 to July 23. By Professor Elliott: Differential equations; Infinite series. By Professor Gergen: Elementary potential theory; Complex variable. By Professor Rankin: Integral calculus; Teaching of mathematics. By Professor Thomas: Modern algebra; Thesis seminar. Second term, July 26 to September 3. By Professor Carlitz: Number theory; Complex variable. By Professor Thomas: Solid analytic geometry; Modern algebra.

University of Illinois. In addition to the usual elementary courses the following advanced courses will be offered: By Professor H. W. Bailey: Advanced aspects of euclidean geometry; Advanced calculus. By Professor D. G. Bourgin: Applications of mathematics. By Dr. O. K. Bower: Elementary statistics—a course for beginners. By Professor A. R. Crathorne: Theory of statistics; Calculus of variations. By Professor Arnold Emch: Theory of equations; Non-euclidean geometry. By Professor H. Levy: Projective geometry; Theory of matrices. By Dr. C. W. Mendel: Complex variable.

University of Kentucky. The following advanced courses will be offered: First term, Projective geometry, by Professor P. P. Boyd, and Theory of equations, by Professor H. H. Downing. Second term, Higher algebra by Professor C. G. Latimer, and Infinite series, by Professor F. Elizabeth LeStourgeon.

University of Iowa. June 11th to August 5th. In addition to courses in college algebra, trigonometry, analytic geometry and calculus, the following advanced courses will be offered: By Dr. Ruth Lane: The teaching of mathematics. By Dr. Edwin Oberg: Differential equations; Fourier and allied series. By Professor L. E. Ward: Advanced calculus, infinite series; Second course in ordinary differential equations. By Professor C. C. Wylie: Mathematics of finance; Astronomy; Readings in astronomy. By Professor J. F. Reilly: Second course in college algebra; Solid analytical geometry; Seminar in finite differences. By Professor E. W. Chittenden: Topics for teachers; Foundations of mathematics; Seminar in analysis. By the staff: Reading and research.

University of Michigan. June 27 to August 19. In addition to elementary courses and the standard courses in differential equations, theory of equations, solid analytic geometery, and advanced calculus, the following courses will be offered: By Professor Carver: Social statistics; Mathematical theory of statistics, II. By Professor Coe: Analytic mechanics; Vector analysis. By Professor A. H. Copeland: Theory of probability. By Professor A. T. Craig: Mathematical theory of statistics, I; Advanced theory of statistics. I. By Professor Dushnik: Empirical formulas. By Dr. Elder: Theory of numbers. By Dr. Friedrichs: Fourier series; Applied mathematics, engineering problems. By Dr. T. N. E. Greville: Finite differences. By Professor T. H. Hildebrandt: Infinite series;

Integral equations. By Professor L. C. Karpinski: Teachers' seminar in algebra; History of geometry and trigonometry. By Mr. Kazarinoff: Synthetic projective geometry. By Professor Norman Miller: Theory of functions of a complex variable. By Dr. Myers: Differential geometry. By Professor J. A. Nyswander: Algebraic theory. By Professor R. L. Wilder: Introduction to the foundations of mathematics; General spaces. In addition, there will be a seminar in pure mathematics conducted by Professors Hildebrandt and Wilder, and one in statistics by Professor Craig.

University of Minnesota. In addition to the usual undergraduate courses, the following courses will be offered: First term, June 15 to July 23. By Professor Dunham Jackson: Advanced algebraic theory; Fourier series. By Professor A. L. Underhill: Intermediate calculus; Differential equations. By Professors Jackson and Underhill: Reading and research in advanced mathematics. Second term, July 25 to August 27. By Professor Gladys Gibbens: Selected topics in advanced mathematics.

University of North Carolina. First term, June 13 to July 23. By Professor Archibald Henderson: Theory of equations; Non-euclidean geometry. By Professor Mackie: Theory of functions of a real variable. By Professor Hill: History of mathematics. By Professor E. T. Browne: Algebraic invariants. Second term, July 25 to August 31. By Professor J. W. Lasley: Synthetic projective geometry. By Professor Hoyle: Advanced calculus; Lie theory of differential equations. By Professor E. A. Cameron: Advanced algebra.

Northwestern University. In addition to courses in analytic geometry and elementary calculus, the following courses will be offered: By Professor F. E. Wood: Series and differential equations; College geometry. By Professor D. R. Curtiss: Determinants and theory of equations; Higher algebra. By Professor H. E. Buchanan: Functions of a complex variable. By Professor A. W. Tucker: Introduction to topology.

Ohio State University. The following courses are announced for the summer of 1938. By Professor C. C. Morris: Finite differences. By Professor Tibor Radó: Projective geometry; Fourier's series. By Professor S. E. Rasor: Introduction to the theory of functions of a complex variable. By Dr. Suckau: Elementary theory of equations; Infinite series and products.

University of Pennsylvania. In addition to the usual undergraduate courses, the following advanced courses will be given: By Professor H. H. Mitchell: Infinite series and products. By Professor Beal: Differential equations; Differential geometry. By Professor P. A. Caris: Diophantine analysis. By Professor J. A. Shohat: Finite differences.

Teachers College, Columbia University. July 5 to August 13. By Professor W. D. Reeve: Mathematics education field course in Germany and England, sailing from New York June 30 with return sailing from Southampton August 19. By Professor C. B. Upton: Reorganization of senior high school mathematics; Teaching arithmetic in primary grades; Teaching arithmetic in intermediate grades. By Professor J. R. Clark: Teaching geometry in secondary schools; Teaching algebra in junior high schools. By Professor C. N. Shuster: Modern business arithmetic; Field work in mathematics. By Dr. J. A. Swenson: Professionalized subject matter in senior high school mathematics; Demonstration class in eleventh year mathematics. By Mr. R. R. Smith: Teaching algebra in secondary schools; Professionalized subject matter in intermediate algebra. By Dr. N. Lazar: Demonstration class in demonstrative geometry; By Dr. A. D. Bradley: Navigation and map projections; Professionalized subject matter in junior high school mathematics.

University of Texas. First term, June 7 to July 18; second term, July 19 to August 29. In addition to the usual elementary courses the following courses will be offered: First term. By Professor A. E. Cooper: Differential equations. By Professor H. V. Craig: Advanced calculus; Vector and tensor analysis. By Professor R. N. Haskell: Potential theory. By Professor P. M. Batchelder: Advanced calculus; Teaching problems. By Professor E. L. Dodd: Probability; Theory of means. By Professor H. J. Ettlinger: Advanced calculus, partial differential equations. By Professor R. L. Moore: Foundations of geometry; Theory of sets. Second term. By Professor C. M. Cleveland: Differential equations. By Professor E. G. Keller: Non-linearity in engineering, physics, and astronomy; Celestial mechanics. By Dr. F. B. Jones: Advanced calculus. By Professor R. G. Lubben: Non-euclidean geometry. By Professor H. S. Vandiver: Foundations of algebra; Modern algebra.

University of Virginia. The following courses in advanced mathematics will be offered: First term, June 20 to July 30. By Professor E. J. McShane: Higher algebra; Calculus of variations. Second term, August 1 to September 3. By Professor G. T. Whyburn: Foundations of geometry; Advanced calculus.

University of Washington. June 20 to August 19. By Professor McFarlan: Integral and differential calculus; Calculus of variations. By Professor Carpenter: Differential equations (with Professor Jerbert); Introduction to applied mathematics. By Professor Cramlet: Vectors and tensors. By Professor Weyl: Selected mathematical problems.

University of Wisconsin. In addition to the usual elementary courses the following courses will be offered: By Dr. Kershner: Theory of equations. By Professor Langer: Theory of analytic functions (six or nine weeks); Fourier's series (six or nine weeks). By Professor March: Theoretical mechanics; Harmonic analysis; Dimensional analysis. By Professor Sokolnikoff; Advanced calculus; Vector and tensor analysis; Definite integrals. By Dr. Wagner: Advanced geometry.

University of Wyoming. In addition to elementary courses, the following will be offered: By Professor O. H. Rechard: Theory of functions of a complex variable (first and second term). By Professor C. F. Barr: College geometry (first term).

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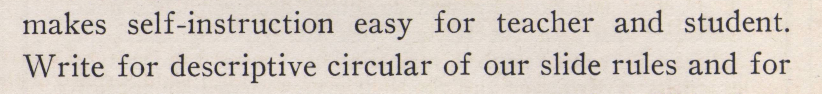

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